PVP 23

Code: 23BS1101

I B.Tech - I Semester – Regular / Supplementary Examinations
DECEMBER 2025

LINEAR ALGEBRA & CALCULUS
(Common for ALL BRANCHES)

Duration: 3 hours Max. Marks: 70

Note: 1. This question paper contains two Parts A and B.
2. Part-A contains 10 short answer questions. Each Question carries 2 Marks.
3. Part-B contains 5 essay questions with an internal choice from each unit. Each Question carries 10 marks.
4. All parts of Question paper must be answered in one place.

BL – Blooms Level CO – Course Outcome

PART – A

		BL	CO
1.a)	Find the rank of the matrix $\begin{bmatrix}1 & 2 & 3\\1 & 4 & 2\\2 & 6 & 5\end{bmatrix}$	L3	CO2
1.b)	Define Echelon and Normal form of a matrix	L2	CO2
1.c)	If a 2×2 matrix A has trace 6 and determinant 8, find its eigen values.	L2	CO2
1.d)	Write the necessary condition for a square matrix to be diagonalizable.	L2	CO2
1.e)	Find c by using Cauchy's mean value theorem for the functions e^x and e^{-x} in the interval [a, b]	L3	CO1
1.f)	State Lagrange's mean value theorem and its geometrical interpretation.	L2	CO1
1.g)	Show that the function $f(x,y) = \frac{x-y}{2x+y}$ is discontinuous at origin.	L2	CO1

1.h)	Estimate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x + y = logz$	L3	CO1
1.i)	Calculate $\int_0^1 \int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}}\, dx\, dy$	L3	CO3
1.j)	Change the double integral from cartesian to polar coordinates $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 + y^2\, dydx$	L3	CO3

PART – B

			BL	CO	Max. Marks
UNIT-I					
2	a)	If $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$, Compute det(AB) directly and verify using Cauchy–Binet formula.	L3	CO1	5 M
	b)	Calculate the value of p if rank of the matrix A is 3 where $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ p & 2 & 2 & 2 \\ 9 & 9 & p & 3 \end{bmatrix}$	L3	CO2	5 M
OR					
3	a)	If the following system has non-zero solution , show that $a + b + c = 0$ or $a = b = c$: $ax + by + cz = 0,\ bx + cy + az = 0,\ cx + ay + bz = 0$	L4	CO4	5 M
	b)	Test for consistency the following equations and solve them if consistent. $x - 2y + 3t = 2,\ 2x + y + z + t = -4,$ $4x - 3y + z + 7t = 8$	L4	CO4	5 M

UNIT-II					
4	If $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ then find A^{-1}, A^4 by using Cayley-Hamilton theorem.		L3	CO2	10 M
OR					
5	Using orthogonal transformation, Reduce the quadratic form $x^2 + 3y^2 + 3z^2 - 2yz$ to canonical form.		L3	CO4	10 M
UNIT-III					
6	a)	Using Rolle's theorem find the value of c for $f(x) = \log\{\frac{x^2+ab}{x(a+b)}\}$ in $[a, b]$	L3	CO5	5 M
	b)	By applying Taylor's series, expand the function $f(x) = \sin x$ about $x = \frac{\pi}{4}$	L3	CO3	5 M
OR					
7	For any $x > 0$, show that $1 + x < e^x < 1 + xe^x$		L3	CO5	10 M
UNIT-IV					
8	a)	If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, Show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$	L3	CO3	5 M
	b)	If $u = x^2 - y^2$, $v = 2xy$ then find $\frac{\partial(u,v)}{\partial(x,y)}$	L3	CO3	5 M
OR					
9	The temperature T at any point in the space (x, y, z) is $T = 400xyz^2$. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = 1$		L4	CO5	10 M

UNIT-V					
10		Evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dydx$ by changing the order of integration.	L3	C05	10 M
OR					
11	a)	Using double integration, find the area between the circles $r = 2sin\theta$ and $r = 4sin\theta$	L4	CO3	5 M
	b)	Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dzdydx$	L3	CO3	5 M

Subject Code: 23BS1101 PVP23

I B.Tech. I Semester Regular Examinations - DECEMBER-2025

LINEAR ALGEBRA and CALCULUS

(Common for ALL Branches)

Key & Scheme of Evaluation

1.a)

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 4 & 2\\ 2 & 6 & 5\end{pmatrix} \quad R_2 \longrightarrow R_2 - R_1, \quad R_3 \longrightarrow R_3 - 2R_1$$ 1

$$\sim \begin{pmatrix}1 & 2 & 3\\ 0 & 2 & -1\\ 0 & 2 & -1\end{pmatrix} \quad R_3 \longrightarrow R_3 - R_2$$

$$\sim \begin{pmatrix}1 & 2 & 3\\ 0 & 2 & -1\\ 0 & 0 & 0\end{pmatrix} \quad \text{so that } \rho(A) = 2$$ 1

Note:-Marks can be awarded for alternate procedure(s) and/or solution(s)

1.b) <u>**Echelon form**</u> A matrix is said to be in echelon form if

i) zero rows, if any, below the non-zero rows

ii) the number of zeros before the first non-zero element in a row is less than the number such zeros in the next row. 1

<u>**Normal form**</u> Every matrix of rank r can be reduced to the form I_r or $(I_r \quad 0)$ or $\begin{pmatrix}I_r & O\\ O & O\end{pmatrix}$ by a finite chain of elementary transformations, where I_r is the unit matrix. The above form is normal form. 1

1.c) $\lambda_1 + \lambda_2 = 6, \ \lambda_1\lambda_2 = 8$

Then $\lambda_1 - \lambda_2 = 2$ 1

$\therefore \lambda_1 = 4, \ \lambda_2 = 2$ 1

1.d) $D = P^{-1}AP$ where $|P| \neq 0$ 1

It has linearly independent eigen vectors (or) 1

The algebraic multiplicity equal to the geometric multiplicity for every eigenvalue of matrix.

1.e) Clearly, $f(x) = e^x, \ g(x) = e^{-x}$ are continuous in $[a, b]$,

$f(x), \ g(x)$ are differentiable in (a, b) where $f'(x) = e^x, \ g'(x) = -e^{-x}$ and $g'(x) \neq 0$ for any $x \in (a, b)$ 1

Then by Cauchy's mean value theorem $\exists \ c \in (a, b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\Rightarrow \frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$\Rightarrow -\frac{e^c}{\left(\frac{1}{e^c}\right)} = \frac{e^b - e^a}{\left(\frac{1}{e^b} - \frac{1}{e^a}\right)}$$

$$\Rightarrow -e^{2c} = -e^a e^b = -e^{a+b}$$

$$\Rightarrow 2c = a + b \Rightarrow c = \frac{a+b}{2} \in (a, b)$$ 1

1.f) If 1) $f(x)$ is continuous in $[a, b]$

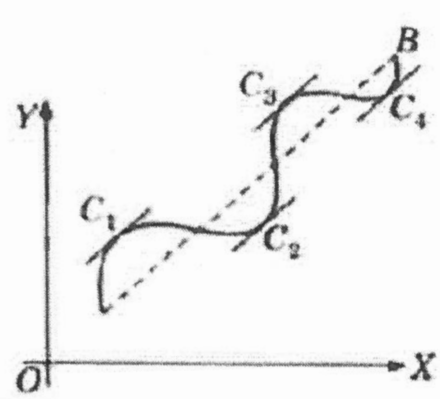
2) $f(x)$ is differentiable in (a, b) then $\exists \ c \in (a, b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$ 1

Geometrical Interpretation. Let A, B be the points on the curve $y = f(x)$ corresponding to $x = a$ and $x = b$ so that $A = [a, f(a)]$ and $B = [b, f(b)]$. (Fig. 4.2)

$\therefore$ Slope of chord $AB = \dfrac{f(b) - f(a)}{b - a}$

By (2), the slope of the chord $AB = f'(c)$, the slope of the tangent of the curve at $C(x = c)$.

Hence the Lagrange's mean value theorem asserts that *if a curve AB has a tangent at each of its points, then there exists at least one point C on this curve, the tangent at which is parallel to the chord AB.* 1



1.g)

$$\lim_{\substack{y\to mx\\ x\to 0}} f(x,y) = \lim_{\substack{y\to mx\\ x\to 0}} \frac{x-y}{2x+y}$$ **1**

$$= lim_{x\to 0}\ \frac{x-mx}{2x+mx}$$

$= \frac{1-m}{2+m}$ which is not unique **1**

Note:-Marks can be awarded for alternate procedure(s) and/or solution(s)

1.h)

$x + y = log\, z \Rightarrow z = e^{x+y}$

$$\frac{\partial z}{\partial x} = e^{x+y}$$ **1**

$$\frac{\partial z}{\partial y} = e^{x+y}$$ **1**

Note:-Marks can be awarded for alternate procedure(s) and/or solution(s)

1.i)

$$\int_0^1\int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}}dxdy$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}}dx \int_0^1 \frac{1}{\sqrt{1-y^2}}dy$$ **1**

$$= [sin^{-1}x\ ]_0^1\,[sin^{-1}y\ ]_0^1 = \frac{\pi^2}{4}$$ **1**

1.j)

$$\int_0^a\int_0^{\sqrt{a^2-x^2}} (x^2+y^2)dxdy = \int_0^{\pi/2}\int_0^a r^3 dr d\theta$$ **2**

2.a)

$$AB = \begin{pmatrix}1 & 2 & 0\\0 & 1 & 3\end{pmatrix}\begin{pmatrix}2 & 0\\1 & 1\\0 & 1\end{pmatrix} = \begin{pmatrix}4 & 2\\1 & 4\end{pmatrix}$$ **2**

Then, $|AB| = \begin{bmatrix}4 & 2\\1 & 4\end{bmatrix} = 14$ **1**

Now, $|A_{12}|\ |B_{12}| + |A_{13}||B_{13}| + |A_{23}||B_{23}| = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\1 & 1\end{bmatrix} + \begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}2 & 0\\1 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$

$= (1)(2) + (3)(2) + (6)(1) = 14$ **1**

$\therefore |AB| = |A_{12}||B_{12}| + |A_{13}||B_{13}| + |A_{23}||B_{23}|$

Hence, Cauchy-Binet formula is verified. **1**

2.b)

$$A = \begin{pmatrix}1 & 1 & -1 & 0\\4 & 4 & -3 & 1\\p & 2 & 2 & 2\\9 & 9 & p & 3\end{pmatrix}\quad R_2 \longrightarrow R_2 - 4R_1,\ R_3 \longrightarrow R_3 - pR_1, R_4 \longrightarrow R_4 - 9R_1$$ **2**

$$\sim\begin{pmatrix}1 & 1 & -1 & 0\\0 & 0 & 1 & 1\\0 & 2-p & 2+p & 2\\0 & 0 & p+9 & 3\end{pmatrix}\quad R_2 \longrightarrow R_2 - 4R_1,\ R_3 \longrightarrow R_3 - pR_1, R_4 \longrightarrow R_4 - 9R_1$$

$$\sim\begin{pmatrix}1 & 1 & -1 & 0\\0 & 2-p & 2+p & 2\\0 & 0 & 1 & 1\\0 & 0 & p+9 & 3\end{pmatrix}\quad R_2 \leftrightarrow R_3$$

$$\sim\begin{pmatrix}1 & 1 & -1 & 0\\0 & 2-p & 2+p & 2\\0 & 0 & 1 & 1\\0 & 0 & 0 & -p-6\end{pmatrix}\quad R_4 \longrightarrow R_4 - (p+9)R_3$$ **2**

Since $\rho(A) = 3,\ |A| = 0 \Rightarrow (2-p)(-p-6) = 0 \Rightarrow p = 2, -6$ **1**

Note:-Marks can be awarded for alternate procedure(s) and/or solution(s)

3.a)

Given equations can be represented in matrix form $AX = 0$

where $A = \begin{pmatrix}a & b & c\\b & c & a\\c & a & b\end{pmatrix}, X = \begin{pmatrix}x\\y\\z\end{pmatrix}, 0 = \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}$ **2**

$AX = O$ has non-zero solution, if $|A| = 0 \Rightarrow \begin{vmatrix}a & b & c\\b & c & a\\c & a & b\end{vmatrix} = 0 \quad R_1 \longrightarrow R_1 + R_2 + R_3$ **1**

$$\Rightarrow \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \quad R_1 \longrightarrow R_1 + R_2 + R_3$$

$$\Rightarrow (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \quad C_2 \longrightarrow C_2 - C_1,\ C_3 \longrightarrow C_3 - C_1$$ **1**

$$\Rightarrow (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c)[(c-b)(b-c) - (a-b)(a-c)] = 0$$

$$\Rightarrow (a+b+c)(bc - c^2 - b^2 - a^2 + ac + ab) = 0$$

$$\Rightarrow (a+b+c)\left[\frac{1}{2}((a-b)^2 + (b-c)^2 + (c-a)^2)\right] = 0$$ **1**

$$\Rightarrow a+b+c = 0 \text{ or } a = b = c$$

3.b)

Given equations can be represented in matrix form $AX = B$

$$\text{where } A = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & 1 & 1 & 1 \\ 4 & -3 & 1 & 7 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix},\ \mathrm{B} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}$$ **2**

$$\text{Augmented matrix, } (A/B) = \begin{pmatrix} 1 & -2 & 0 & 3 & 2 \\ 2 & 1 & 1 & 1 & -4 \\ 4 & -3 & 1 & 7 & 8 \end{pmatrix} \quad R_2 \longrightarrow R_2 - 2R_1,\ R_3 \longrightarrow R_3 - 4R_1$$

$$\sim \begin{pmatrix} 1 & -2 & 0 & 3 & 2 \\ 0 & 5 & 1 & -5 & -8 \\ 0 & 5 & 1 & -5 & 0 \end{pmatrix} \quad R_3 \longrightarrow R_3 - R_2$$ **2**

$$\sim \begin{pmatrix} 1 & -2 & 0 & 3 & 2 \\ 0 & 5 & 1 & -5 & -8 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}$$

Here $\rho(A/B) = 3 \neq \rho(A) = 2$ **1**

$\therefore$ The system is inconsistent.

4)

The characteristic equation of A is $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - a\lambda^2 + b\lambda - c = 0$ where $a = 4 + 3 - 3 = 4$ **1**

$$b = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ -1 & -3 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix} = -1 - 6 + 6 = -1$$

$$c = |A| = \begin{vmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{vmatrix} = -4$$ **2**

$\therefore \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$ **1**

By Cayley-Hamilton theorem, we have $A^3 - 4A^2 - A + 4I = 0$ **1**

$$A^2 = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix}\begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix} = \begin{pmatrix} 16 & 18 & 18 \\ 5 & 7 & 6 \\ -5 & -6 & -5 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 16 & 18 & 18 \\ 5 & 7 & 6 \\ -5 & -6 & -5 \end{pmatrix}\begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix} = \begin{pmatrix} 64 & 78 & 78 \\ 21 & 27 & 26 \\ -21 & -28 & -27 \end{pmatrix}$$ **3**

$$A^4 = 4A^3 + A^2 - 4A = \begin{pmatrix} 256 & 306 & 306 \\ 85 & 103 & 102 \\ -85 & -102 & -101 \end{pmatrix}$$ **1**

$$A^{-1} = \frac{1}{4}(-A^2 + 4A + I) = \frac{1}{4}\begin{pmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{pmatrix}$$ **1**

5)

Let $X'AX = x^2 + 3y^2 + 3z^2 - 2yz$ where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ **2**

The characteristic equation of A is $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$ which gives **2**

$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

$$\Rightarrow \lambda = 1, 2, 4$$ **1**

Eigen value	1	2	4
Eigen vector	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

3

Modal matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ **1**

Normalized modal matrix, $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

Then $P'AP = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

the canonical form is given by $Y'DY = y_1^2 + 2y_2^2 + 4y_3^2$ where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = PY = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$ **1**

6.a)

$f(x) = log\left[\frac{x^2+ab}{x(a+b)}\right] = log(x^2+ab) - log\,x - log(a+b)$ in $[a,b]$

Then clearly, $f(x)$ is continuous and differentiable. **2**

Also, $f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$ and $f(a) = log\left[\frac{a^2+ab}{a(a+b)}\right] = \log 1 = 0$

$$f(b) = log\left[\frac{b^2+ab}{b(a+b)}\right] = \log 1 = 0 = f(a)$$ **1**

By Rolle's mean value theorem, $\exists\, c \in (a,b) \ni f'(c) = 0$

$$\Rightarrow \frac{2c}{c^2+ab} - \frac{1}{c} = 0$$ **1**

$$\Rightarrow c^2 - ab = 0$$

$$\Rightarrow c = \pm\sqrt{ab}$$

The value $c = \sqrt{ab} \in (a,b)$. Hence, Rolle's theorem is verified. **1**

6.b)

$f(x) = sin\,x$ $\qquad f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f'(x) = cos\,x$ $\qquad f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f''(x) = -\,sin\,x$ $\qquad f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ **2**

Substituting these values in the Taylor's series

$$f(x) = f\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)}{1!} f'\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \cdots\cdots$$ **2**

$$= \frac{1}{\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)}{1!}\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \cdots\cdots$$

$$= \frac{1}{\sqrt{2}}\left[1 + \frac{\left(x-\frac{\pi}{4}\right)}{1!} - \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} + \cdots\cdots\right]$$ **1**

7)

Let $f(x) = e^x$ in $[0, x]$

Then clearly, $f(x)$ is continuous and differentiable. **2**

Also, $f'(x) = e^x$

By Lagrange's mean value theorem, $\exists\, c \in (0, x) \ni f'(c) = \frac{f(x)-f(0)}{x-0}$ **2**

$$\Rightarrow e^c = \frac{e^x - 1}{x}$$

Now, $c \in (0,x) \Rightarrow 0 < c < x$ **2**

$$\Rightarrow 1 < e^c < e^x$$

$$\Rightarrow 1 < \frac{e^x - 1}{x} < e^x$$ **2**

$$\Rightarrow x < e^x - 1 < xe^x$$ **2**

$$\Rightarrow 1 + x < e^x < 1 + xe^x$$

Note:- Marks can be awarded for alternate procedure(s) or method(s) or solution(s).

8.a)

$u = log(x^3 + y^3 + z^3 - 3xyz) \quad \longrightarrow \boxed{1}$

Differentiating $\boxed{1}$ partially with respect to x, we get $\frac{\partial u}{\partial x} = \frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}$

Similarly, $\frac{\partial u}{\partial y} = \frac{3(y^2-xz)}{x^3+y^3+z^3-3xyz}$, $\frac{\partial u}{\partial z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}$ **3**

$$\therefore \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}+\frac{3(y^2-xz)}{x^3+y^3+z^3-3xyz}+\frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}\right)$$ **1**

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3(x^2+y^2+z^2-xy-yz-xz)}{(x+y+z)\,(x^2+y^2+z^2-xy-yz-xz)}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}$$ **1**

Note:- Marks can be awarded for alternate procedure(s) or method(s) or solution(s).

8.b)

$u = x^2 - y^2, \ v = 2xy$

Then $u_x = 2x, \ u_y = -2y, \ v_x = 2y, \ v_y = 2x$ **3**

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2)$$ **2**

Note:- Marks can be awarded for alternate procedure(s) or method(s) or solution(s).

9)

Let $F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z) = 400xyz^2 + \lambda\,(x^2+y^2+z^2-1)$

Then, $\frac{\partial F}{\partial x} = 400yz^2 + \lambda\,(2x) = 0$

$$\Rightarrow 400yz^2 = -\lambda\,(2x) \Rightarrow -\frac{400\,xyz^2}{2\lambda} = x^2 \longrightarrow \boxed{1}$$ **2**

$$\frac{\partial F}{\partial y} = 400xz^2 + \lambda\,(2y) = 0 = 0$$

$$\Rightarrow 400xz^2 = -\lambda\,(2y) \Rightarrow -\frac{400\,xyz^2}{2\lambda} = y^2 \longrightarrow \boxed{2}$$ **2**

$$\frac{\partial F}{\partial z} = 800xyz + \lambda\,(2z) = 0$$

$$\Rightarrow 800xyz = -\lambda\,(2z) \Rightarrow -\frac{800\,xyz^2}{2\lambda} = z^2 \longrightarrow \boxed{3}$$ **2**

From, $\boxed{1}$, $\boxed{2}$ & $\boxed{3}$ we have $x^2 = y^2 = \frac{z^2}{2}$ **1**

$\therefore x^2 + y^2 + z^2 = 1$

$$\Rightarrow \frac{z^2}{2}+\frac{z^2}{2}+z^2=1$$
$$\Rightarrow 2z^2=1$$
$$\Rightarrow z=\frac{1}{\sqrt{2}} \quad \text{then } x=y=\frac{1}{2}$$ **2**

Hence, the highest temperature on the surface is $400xyz^2 = 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)^2 = 50$ **1**

Note:- Marks can be awarded for alternate procedure(s) or method(s) or solution(s).

10) Here first order of integration w.r.t. y from x to $\sqrt{2-x^2}$ along the vertical strip (i.e. $y=x$ to $x^2+y^2=2$) and then integration is done w.r.t. x from 0 to 1 as shown in the figure. **3**

By, changing the order of integration,

$$I=\int_{y=0}^{1}\int_{x=0}^{y}\frac{x}{\sqrt{x^2+y^2}}dxdy+\int_{y=1}^{\sqrt{2}}\int_{x=0}^{\sqrt{2-y^2}}\frac{x}{\sqrt{x^2+y^2}}dxdy$$ **3**

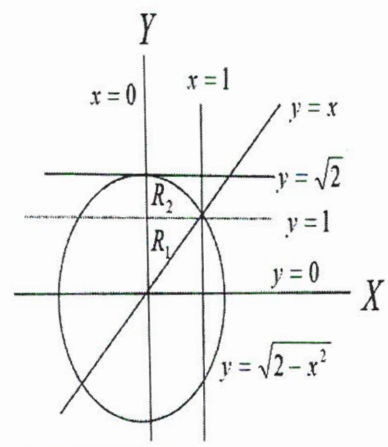
$$=\frac{1}{2}\int_{y=0}^{1}\int_{x=0}^{y}2x(x^2+y^2)^{\frac{-1}{2}}dxdy+\frac{1}{2}\int_{y=1}^{\sqrt{2}}\int_{x=0}^{\sqrt{2-y^2}}2x(x^2+y^2)^{\frac{-1}{2}}dxdy$$

$$=\int_{y=0}^{1}[\sqrt{x^2+y^2}]_0^y dy+\int_{y=1}^{\sqrt{2}}[\sqrt{x^2+y^2}]_0^{\sqrt{2-y^2}}dy$$

$$=\int_{y=0}^{1}[\sqrt{2}\,y-y]dy+\int_{y=1}^{\sqrt{2}}[\sqrt{2}-y]dy$$ **3**

$$=(\sqrt{2}-1)[\frac{y^2}{2}]_0^1+[\sqrt{2}\,y-\frac{y^2}{2}]_1^{\sqrt{2}}$$

$$=\frac{\sqrt{2}}{2}-\frac{1}{2}+2-1-\sqrt{2}+\frac{1}{2}=1-\frac{\sqrt{2}}{2}=1-\frac{1}{\sqrt{2}}$$ **1**



11.a)
$$\iint r\,drd\theta=\int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r dr d\theta$$

$$=\int_0^{\pi}\left[\frac{r^2}{2}\right]_{2\sin\theta}^{4\sin\theta}d\theta$$ **2**

$$=\int_0^{\pi}\left[\frac{16\sin^2\theta-4\sin^2\theta}{2}\right]d\theta$$ **1**

$$=\int_0^{\pi}6\sin^2\theta\,d\theta$$ **1**

$$=6\cdot 2\int_0^{\pi/2}\sin^2\theta\,d\theta=12\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)=3\pi$$ **1**

11.b)
$$\int_0^a\int_0^x\int_0^{x+y}e^{x+y+z}\,dz\,dy\,dx=\int_0^a\int_0^x e^{x+y}\left[e^z\right]_0^{x+y}dy\,dx$$

$$=\int_0^a\int_0^x e^{x+y}[e^{x+y}-1]\,dydx$$ **2**

$$=\int_0^a\int_0^x (e^{2x+2y}-e^{x+y})\,dydx$$

$$=\int_0^a e^{2x}\left[\frac{e^{2y}}{2}\right]_0^x-e^x[e^y]_0^x\,dx$$

$$=\int_0^a\left[e^{2x}\left(\frac{e^{2x}}{2}-\frac{1}{2}\right)-e^x(e^x-1)\right]dx$$ **2**

$$=\int_0^a\frac{1}{2}(e^{4x}-3e^{2x}+2e^x)\,dx$$

$$=\frac{1}{2}\left[\frac{e^{4x}}{4}-3\frac{e^{2x}}{2}+2e^x\right]_0^a=\frac{e^{4a}}{8}-\frac{3e^{2a}}{4}-e^a+\frac{3}{8}$$ **1**

Note:- Marks can be awarded for alternate procedure(s) or method(s) and/or solution(s)