Relations

Introduction :-

Human language has many words & phrases to describe relationships between or among objects. It may be that for two people A & B; that A is a Parent of B, A is taller than B, A is infront of B etc. In, algebra, it may be that the value of variable x is less than the value of variable y. In set theory it may be that a set "x" is a subset of a set "y" (or) that x is disjoint from y.

Suppose a<b, we can say that there is a relation between a & b, we say that "a is related to b". The relation is "less than". If we from the related elements as an ordered Pair i.e (a,b). these types of elements are the elements of a Cartesian Product A×B.

A×B is read as "A cross B"

Cartesian Product :- suppose A & B are any two non empty sets then the Cartesian Product of A & B is denoted by "A×B" & It is defined as

$$A\times B = \{(x,y) / x\in A \ \& \ y\in B\}$$

If the set A consists of "m" Elements and the set B consists of 'n' Elements, then we can form 'mn' relation from A to B.

The Relation "R" is the ~~set~~ subset of the Cartesian Product $A \times B$.

Problem. Suppose $A = \{1,2,3,4\}$ & $B = \{3,4,6,7,8\}$ then $A \times B$ consists of 20 elements.

let $R_1 = \{(1,3)(3,6)(2,6)(3,8)\}$

$R_2 = \{(1,4)(1,6)(2,4)(3,6)(3,7)(4,7)(4,8)\}$

$R_3 = \{(1,3)(1,4)(2,3)(2,7)(3,8)(4,8)\}$

R_1, R_2, R_3 are different relations.

In R_1, 1 is related to 3
3 is related to 6
2 " " " 6
3 " " " 8

Hly In R_2 & R_3.

Domain :- The set $\{a \in A / aRb$ for some $b \in B\}$ is called the domain of R.

Range : The set $\{b \in B / aRb$ for some $a \in A\}$ is called the range of R.

or R_1, the domain $= (1,2,3)$ & range $= (3,6,8)$.
R_2 " " $= (1,2,3,4)$ & range $= (4,6,7,8)$
R_3 " " $= (1,2,3,4)$ & range $= (3,4,7,8)$

Note. Domain of $R \subseteq A$, Range of $R \subseteq B$

Problem. Let $A = \{2,3,4\}$ & $B = \{3,4,5,6,7\}$

Define a relation R from A to B by $(a,b) \in R$ if
~~divisible b~~ "a divides b" i.e $(a|b)$

Then $R = \{(2,4)(2,6)(3,3)(3,6)(4,4)\}$

Hence domain of R is $\{2,3,4\}$ &
range of R is $\{3,4,6\}$

Definition :- Let "R" be a relation from A to B. Then inverse of relation R from B to A is denoted by R^{-1} & it is defined as $R^{-1} = \{(b,a) / (a,b) \in R\}$

Problem. If $R = \{(2,4)(2,6)(3,3)(3,6)(4,4)\}$ is a relation from A to B.

Then $R^{-1} = \{(4,2)(6,2)(3,3)(6,3)(4,4)\}$

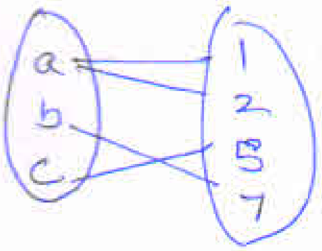
is inverse relation from B to A.

The relations by Venn diagram

Suppose $A = (a, b, c)$ & $B = (1, 2, 5, 7)$

If there is a relation, then there is a line connecting the element of A to the element of B.

If $R = \{(a,1)\ (a,2)\ (b,7)\ (c,5)\}$.



Properties of Relations

A relation "R" on a set A is said to be

(i) Reflexive on A if $(a,a) \in R$ (ie) $aRa \ \forall a \in A$

(ii) Symmetric on A if $(a,b) \in R \Rightarrow (b,a) \in R \ \forall a,b \in A$

(iii) Transitive on A if $(a,b) \in R$, $(b,c) \in R$ $\Rightarrow (a,c) \in R \ \forall a,b,c \in A$.

Note

1) If R is reflexive then "R" is reflexive.

(2) The symbol "=" is symmetric. If $a = b$ then $b = a$.

3. the symbol "$<$" is not Symmetric.

If $a<b$, then b is not less than a.

A is a son of B is not Symmetric

($\because$ B Cannot be the son of A)

Problem: Let $A=\{1,2,3,4\}$ & $R=\{(1,1)(2,2)(3,3)(4,4)(2,3)(3,1)(2,1)\}$.

is reflexive & transitive.

$\because (2,3) \in R,\ (3,1) \in R \Rightarrow (2,1) \in R$.

$\because (3,2) \notin R$ & $(1,3) \notin R,\ (1,2) \notin R$.

(2) Let $A=\{(1,2,3,4)\}$ & $R=\{(1,3)(4,2)(2,4)(2,3)(3,1)\}$

R is not symmetric because $(2,3) \in R$ but $(3,2) \notin R$.

R is not reflexive.

R is not transitive.

$\because (2,3) \in R, (3,1) \in R$ but $(2,1) \notin R$.

(3) If $a<b$ & $b<c$ then $a<c$, R is transitive.

Anti-symmetric:- A relation R on a set A is called anti-symmetric if $(a,b) \in R$ & $(b,a) \in R$ then $a=b$ (or) if $a R b$, $b R a \Rightarrow a=b$ $\forall a,b \in A$.

A relation R on a set A is said to be

(i) irreflexive if $(a,a) \notin R$ i.e a is not related to a. $\forall a \in A$

(ii) Anti-symmetric on A if aRb, $bRa \Rightarrow a=b$ $\forall a,b \in A$.

(iii) As symmetric on A if $aRb \Rightarrow b$ is not related to a. i.e $(a,b) \in R \Rightarrow (b,a) \notin R$ $\forall a,b \in A$.

Problem: Given an example of relation which is symmetric but neither reflexive nor anti-symmetric nor transitive.

Sol: Let $A = \{1,2,3\}$

Consider the relation $R = \{(1,1)(1,2)(2,1)(3,2)(2,3)\}$ on A.

(i) R is symmetric

$\because (1,2) \in R \Rightarrow (2,1) \in R$.

$(3,2) \in R \Rightarrow (2,3) \in R$.

$(1,1) \in R \Rightarrow (1,1) \in R$.

Then R is symmetric on A.

(ii) R is not reflexive.

$\because (2,2) \notin R$, $(3,3) \notin R$

$\Rightarrow$ R is not reflexive.

(iii) R is not anti- Symmetric.

$\because (1,2) \in R,\ (2,1) \in R \Rightarrow 1 \neq 2.$

$\Rightarrow$ R is not anti-Symmetric relation on A.

(iv) R is not transitive.

$(1,2) \in R,\ (2,3) \in R \Rightarrow (1,3) \notin R.$

$\therefore R$ is not transitive on A.

<u>Problem</u>: Given an Example of a relation which is transitive. but neither reflexive nor symmetric nor antisymmetric.

<u>Sol</u>: Let $A = \{1,2,3\}$.

$R = \{(1,1)\,(2,2)\,(1,2)(1,3)\,(2,1)\,(2,3)\}$ be relation on A.

$\because (3,3) \notin R$ is not reflexive.

$\because (1,3) \in R$ but $(3,1) \notin R$ $\therefore R$ is not symmetric

$\because (1,2) \in R$ & $(2,1) \in R$ but $1 \neq 2$.

$\Rightarrow$ R is not anti-symmetric

$\because (1,2) \in R$ & $(2,3) \in R \Rightarrow (1,3) \in R.$

$\Rightarrow$ R is transitive on A.

Problem Give an Example of a relation which is anti-symmetric but neither reflexive nor symmetric nor transitive.

Sol:- Let $R = \{(x,y) \in R \times R, \because x,y$ are integers & $x+3y=12\}$

$\because (1,1) \notin R$, R is not reflexive.

$(6,2) \in R$ but $(2,6) \notin R \Rightarrow$ R is not symmetric.

Let $(x,y), (y,x) \in R$

Then $x+3y=12 = y+3x$.

$\Rightarrow 2x = 2y$

$\Rightarrow x = y$

Hence R is anti-symmetric

& R is not transitive, because $(-6,6)\ (6,2) \in R$ but $(-6,2) \notin R$.

Problem: Let $A = \{1,2,3,4\}$, Relation $R = \{(1,2)\ (2,4)$ is not reflexive, not symmetric & not transitive on A.

Problem. Let $A = \{1,2,3,4\}$. Relation $R = \{(1,1)(2,2)(3,3)(4,4)(2,3)(3,1)(2,1)\}$ is reflexive & transitive, but not symmetric on A.

Problem. Let $A = \{1,2,3,4\}$ & relation $R = \{(1,3)(2,3)(2,4)(3,1)(4,2)\}$ then the relation R is irreflexive, as symmetric not transitive & not anti-symmetric on A.

Equivalence Relations

A relation R on a set A is said to be an Equivalence relation on A if

(i) R is reflexive on A

(ii) R is symmetric on A

(iii) R is Transitive on A.

Problem: Let R be a relation on N defined as

$R = \{(a,b) \mid (a,b) \in N \text{ \& } (a+b) \text{ is Even}\}$. Then R is an Equivalence relation on N.

(i) $a+a$ is Even for any $a \in N$.

(ii) If $a+b$ is Even $\Rightarrow b+a$ is also Even $a,b \in N$.

(iii) If $a+b$ is Even, $b+c$ is Even.

$\Rightarrow a+b+b+c = a+c$ is also Even $\forall a,b,c \in N$.

Problem: Show that the relation R is defined $N \times N$ by $(a,b) \, R \, (c,d)$ iff $a+d = b+c$ is an Equivalence relation.

Sol: $N \times N = \{(a,b) \mid a,b \in N\}$.

(i) $(a,b) R (a,b) \;\forall\; (a,b) \in N \times N$.

$\therefore a+d = b+c$.

R is reflexive on $N \times N$.

(ii) Let $(a,b)\ R\ (c,d)\ \forall\ (a,b),(c,d) \in N\times N$

$\Rightarrow a+d = b+c$

$\Rightarrow c+b = d+a.$

$\Rightarrow (c,d)\ R\ (a,b)$

Hence R is symmetric on $N\times N$

(iii) Let $(a,b)\ R\ (c,d), (c,d)\ R\ (e,f)$

$\forall (a,b),(c,d),(e,f) \in N\times N$

$\Rightarrow a+d = b+c$ & $c+f = d+e.$

$\Rightarrow a+d+c+f = b+c+d+e.$

$\Rightarrow a+f = b+e$

$\Rightarrow (a,b)\ R\ (e,f).$

$\Rightarrow$ R is transitive on $N\times N.$

Hence R is an Equivalence relation on $N\times N.$

Problem:- Let R be the relation on $N\times N$ which is defined by $(a,b)\ R\ (c,d)$ which can be written as $(a,b)\ R\ (c,d)$ iff $ad = bc$. P.T R is an Equivalence Relation.

Sol:- $N\times N = \{(a,b) \mid a,b \in N\}$

(i) $(a,b)\ R\ (a,b)\ \forall (a,b) \in N\times N.$

$\Rightarrow ab = ba.$

$\therefore$ R is reflexive on $N\times N.$

(ii) Let $(a,b) R (c,d)$ $\forall (a,b), (c,d) \in N \times N$.

$\Rightarrow ad = bc$

$\Rightarrow cb = da$

$\Rightarrow (c,d) R (a,b)$

$\Rightarrow$ R is Symmetric on $N \times N$.

(iii) Let $(a,b) R (c,d)$ & $(c,d) R (e,f)$ $\forall (a,b), (c,d) (e,f) \in N \times N$

$\because (a,b) R (c,d) \Rightarrow ad = bc$

$(c,d) R (e,f) \Rightarrow cf = de$

$\Rightarrow (ad)(cf) = (bc)(de)$

$a(dc)f = b(cd)e$

$a(cd)f = b(cd)e$ $[dc = cd]$

$\Rightarrow af = be$ by Cancelling cd on both sides.

$\Rightarrow (a,b) R (e,f)$

$\Rightarrow$ R is transitive on $N \times N$.

$\therefore$ It Satisfied all 3 Properties

$\therefore$ R is Equivalence relation on $N \times N$.

Problem: If R is a relation on the set of integers Z defined by $R = \{(x,y) / (x-y)$ is divisible by $3\}$ then Prove that R is an Equivalence relation.

Solution:

Let us define $R = \{(x,y) / (x-y)$ is divisible by $3\}$

(i) For any $x \in X$, $x - x = 0$ is divisible by 3.

$x R x$

$\Rightarrow$ R is reflexive.

(ii) For any $x, y \in X$

let $x R y$ then $x - y$ is divisible by 3.

$\Rightarrow$ $y - x$ is also divisible by 3.

$\Rightarrow$ $y R x$.

Hence $x R y \Rightarrow y R x$.

$\Rightarrow$ the relation R is symmetric.

(iii) For any $x, y, z \in X$.

let $x R y$ & $y R z$ then $(x-y)$ & $(y-z)$ are divisible by 3.

$\Rightarrow (x-y) + (y-z) = x - z$ is also divisible by 3

$\Rightarrow x R z$.

$\Rightarrow$ the relation R is transitive.

$\therefore$ The 3 properties are satisfied.

$\therefore$ The relation R is an Equivalence relation.

<u>Problem</u>: Given $S = \{1, 2, 3, \dots 10\}$ & a relation R on S where $R = \{(x, y) / x + y = 10\}$. what are the properties of the relation R ?

Solution :-

Given $R = \{ (x,y) / x+y=10 \}$

i.e $R = \{ (1,9) (2,8) (3,7) (4,6) (5,5) (6,4)$
$(7,3) (8,2) (9,1) \}$

(i) For any $x \in X$ $\forall$ $(x,x) \notin R$.

Here, $1 \in X$ but $(1,1) \notin R$.

$\Rightarrow$ the relation R is not reflexive. but it is irreflexive.

(ii) $(1,9) \in R \Rightarrow (9,1) \in R$.

$(2,8) \in R$, $(8,2) \in R$.

Hence if $(x,y) \in R$ then $(y,x) \in R$ $\forall$ $(x,y) \in X$.

$\Rightarrow$ The relation R is symmetric, but it is not antisymmetric.

(iii) $(1,9) \in R$ & $(9,1) \in R$.

$\Rightarrow (1,1) \notin R$.

Thus, if $(x,y) \in R$ & $(y,z) \in R$ then $(x,z) \notin R$.

$\Rightarrow$ The relation R is not transitive.

Hence R is irreflexive & Symmetric.

Problem: Given an Example of a relation that is neither reflexive nor irreflexive.

Sol. Let $X = \{1,2,3\}$

Let us consider the relation $R = \{(1,1)\ (1,2)\ (3,2)\ (2,3)\ (3,3)\}$

$\Rightarrow$ the Relation R is neither reflexive nor irreflexive

[$\because (2,2) \notin R$ it is not reflexive &
$(1,1) \in R$ & $(3,3) \in R$.
$\therefore$ It is irreflexive]

Representation of Relations

There are two methods of representation of the relation.

(i) Matrix method

(ii) Direct graph method.

Matrix method.

A binary relation (finite relation) R from a set 'A' with 'n' elements to a set 'B' with "m" elements is represented by "$n \times m$" matrix called "Relation matrix" denoted by $M_R = [a_{ij}]$.

where $a_{ij} = 1$ if $(a_i, b_j) \in R$
$= 0$ if $(a_i, b_j) \notin R$.

$(a_i, b_j) \in R$ means i^{th} element of A is related to the j^{th} element of B.

Problem: Let $A = \{a, b, c\}$ $B = \{1, 2, 3, 4\}$

$R: A \to B$ as $R = \{(a,1)(a,3)(b,2)(b,4)(c,2)(c,3)\}$

$$M_R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}_{3 \times 4} \end{array}$$

$(a,1) \in R$ [So first row & First Column Element is 1]

$(a,2) \notin R$ [" " " 2nd " is 0]

The Relation Matrix M_R is also called as the 'Boolean Matrix'

Problem: Let $A = \{1,2,3,4,5,6\}$. Define Relation R as less than on A then find Relation Matrix on A

Sol: Given relation is less than $R : A \to A$.

where $A = \{1,2,3,4,5,6\}$

then $R = \{(1,2)(1,3)(1,4)(1,5)(1,6)(2,3)(2,4)(2,5)(2,6)(3,4)(3,5)(3,6)(4,5)(4,6)(5,6)]$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Problem:- Let $A = \{a_1, a_2\}$ $B = \{b_1, b_2, b_3, b_4, b_5\}$

and Relation Matrix $M_R = \begin{array}{c|c} & \begin{matrix} b_1 & b_2 & b_3 & b_4 & b_5 \end{matrix} \\ \begin{matrix} a_1 \\ a_2 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$

then write the relation.

$R = \{(a_1, b_2)(a_1, b_4)(a_2, b_1)(a_2, b_3), (a_2, b_4)\}$

Note:- The matrix M_R has the elements as 1 & 0.

Properties

(i) R is reflexive iff all the elements in the main diagonal of M_R are Equal to 1.

(ii) R is Symmetric if $a_{ij} = a_{ji}$ $\forall i, j$ i.e

$M_R = (M_R)^T$

(iii) R is anti-Symmetric if $a_{ij} = 1$ with $i \neq j$ then $a_{ji} = 0$

In otherwords Either $a_{ij} = 0$ (or) $(a_{ji}) = 0$ when $i \neq j$

Problem. Suppose the relation R on a set is represented by matrix $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is R reflexive, Symmetric, not anti-symmetric.

(i) $\because$ all the diagonal Elements of the matrix are Equal to 1

$\therefore$ R is reflexive.

(ii) $\therefore M_R = (M_R)^T$, R is Symmetric.

(iii) R is not anti-symmetric.

Problem: Let $A = \{1,2,3,4\}$ & $B = \{b_1, b_2, b_3\}$

Consider the relation $R = \{(1,b_2)(1,b_3)(3,b_2)(4,b_1)(4,b_3)\}$

Determine the Matrix of the relation.

Sol: Given $A = \{1,2,3,4\}$ $B = \{b_1, b_2, b_3\}$

$R = \{(1,b_2)(1,b_3)(3,b_2)(4,b_1)(4,b_3)\}$

(M_R) Matrix of the relation R is written as

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

Prob

Let $A = \{1,2,3,4\}$. Find the relation R on A determined by the Matrix

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Solution :- Given $M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$

The Relation $R = \{(1,1)(1,3)(2,3)(3,1)(4,1)(4,2)(4,4)\}$

Directed Graph Method (Digraph).

Suppose R is relation on set 'A' where $A = \{a_1, a_2, \dots a_n\}$. The elements of A are represented by points (or) circles called nodes (or) vertices of the graph. An arrow is drawn from the vertex a_i to a_j iff $(a_i, a_j) \in R$. This is called an (directed) edge. This pictorial representation of "R" is called a directed graph (or) digraph of R.

An element of the form (a, a) in a relation corresponds to a directed edge from a to a such an edge is called a loop

(or)

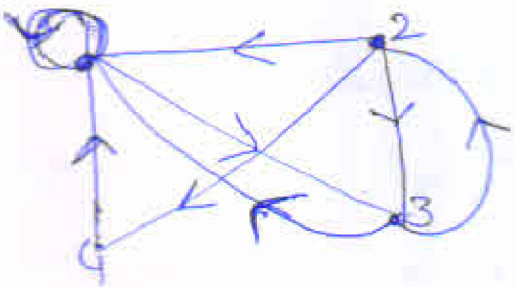
An edge from a vertex to itself is called a loop.

Prob: Draw the graph for the following relations:

(i) $R = \{(1,1)\ (2,2)\ (1,2)\}$ on $X = \{1, 2\}$

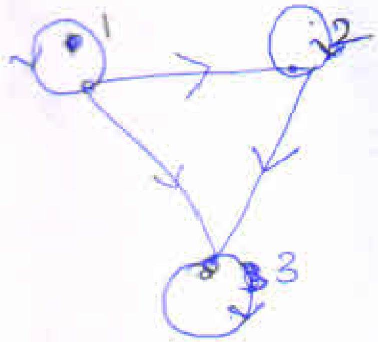
(ii) $S = \{(1,1)\ (1,3)\ (2,1)\ (2,3)\ (2,4)\ (3,1)\ (3,2)\ (4,1)\}$

on a set $X = \{1,2,3,4\}$



(iii) $R = \{(1,1)\ (1,2)\ (1,3)\ (2,2)\ (2,3)\ (3,3)$ on

$Y = \{1,2,3\}$



Prob

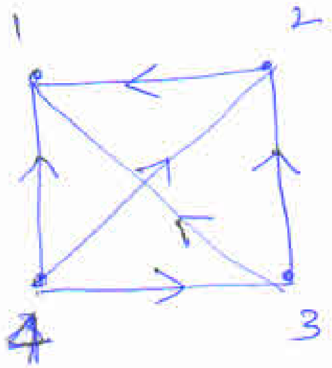
Let $X = \{1,2,3,4\}$ & relation $R = \{(x,y) \mid x>y\}$. Draw the graph of R & also given its matrix

Given $X = \{1,2,3,4\}$

$R = \{(x,y) \mid x>y\}$

$= \{(2,1)\ (3,1)\ (3,2)\ (4,1)(4,2)(4,3)\}$

Graph of R.



Matrix of R.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Properties of relations.

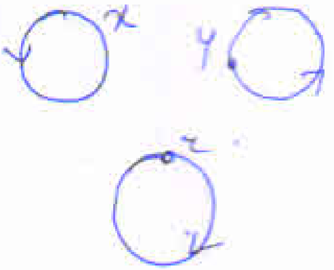
1) If a relation is reflexive, than there is a loop at Every Point.

2) If a relation is symmetric & if one Point is Connected to another, than there must be a ~~relation~~ return arc from the Second Point to the Point

(3) for anti symmetric relations, no direct relation Paths should Exist.

(4) If a relation is transitive. then the Situation is not So Simple.

A diagraph is reflexive if every vertex has an edge from the vertex itself (self loop)



$\forall x,y,z \quad xRx,\ yRy,\ zRz.$

A diagraph is irreflexive if none of the vertices have self-loops

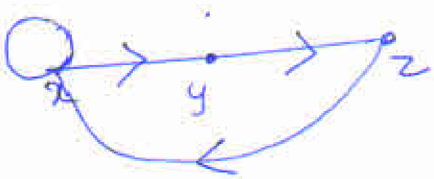
$\forall x,y,z, \quad x \not R x.$

irreflexive.

A diagraph is symmetric if for every edge in one direction between points there is also an edge in the opposite direction between the same two points

Symmetric.

A diagram is Anti-symmetric if no two distinct points have an edge going between them in both directions



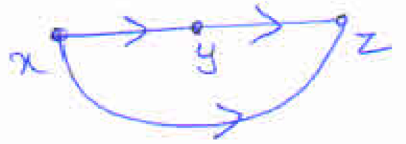
$\forall a,b \in A$

if $(a,b) \in R \Rightarrow (b,a) \in R$.

unless $a=b$.

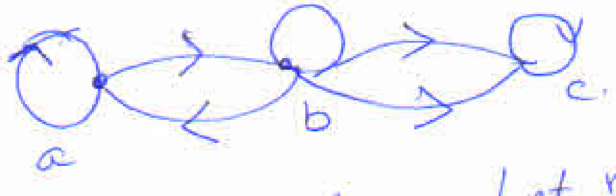
If R is an anti-symmetric relation then for different vertices i & J there connect be edge from vertex i to vertex J & an edge from vertex J to vertex i.

A diagraph is <u>transitive</u> if for any 3 vertices x, y, z whenever there is an edge from x to y & an edge from y to z there is also an edge from x to z.

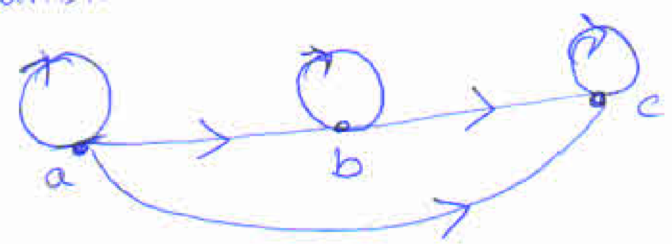


<u>Problem</u>. Given an example of a nonempty set and a relation on the set that satisfies each of the following combinations of properties, draw a diagraph of the relation.

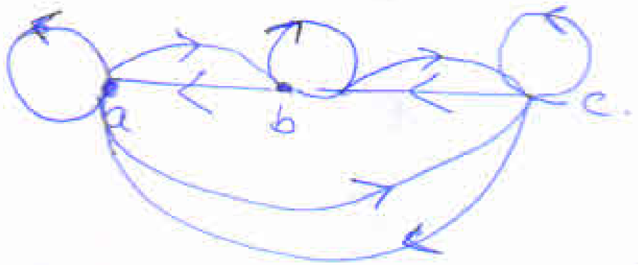
1) symmetric & reflexive but not transitive.



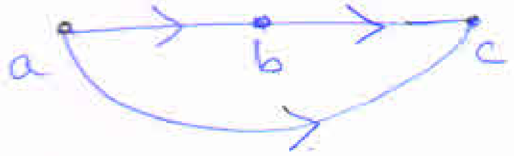
(2) Transitive & reflexive but not symmetric



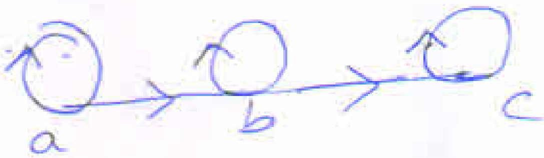
(3) Transitive & reflexive, symmetric but not anti-sym



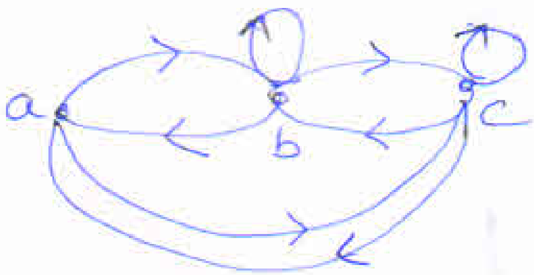
(4) Transitive & anti-symmetric but not reflexive.



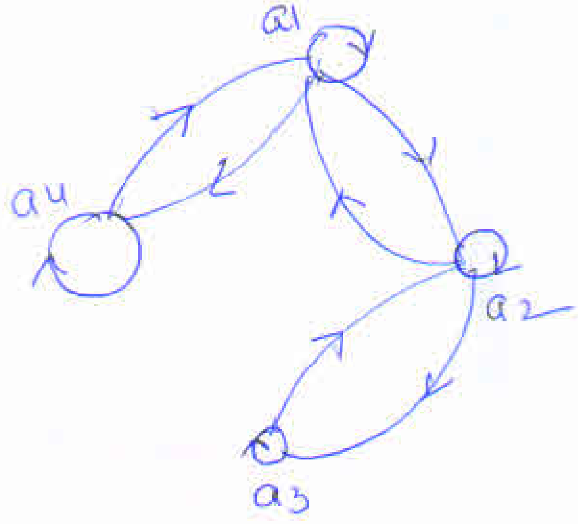
(5) Anti-symmetric & reflexive but not transitive.



(6) Symmetric, transitive but not reflexive.



5b Find the relation determined by the given graph & the corresponding relation matrix. Also determine the properties of the relation given by the graphs.



Solution:- $R = \{(a_1, a_1), (a_1, a_2)\ (a_2, a_1)\ (a_2, a_2)\ (a_2, a_3)\ (a_3, a_2)$
$(a_3, a_3)\ (a_1, a_4)\ (a_4, a_1)\ (a_4, a_4)\}$ is

the corresponding Matrix of the relation is

written as

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

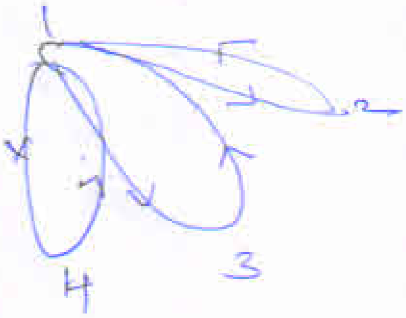
The relation is reflexive [∵ Every vertex has a loops]

& symmetric [∵ whenever $a_i R a_j$ then $a_j R a_i$] i.e

$$M_R = (M_R)^T$$

The relation is not transitive.

Prob



$R = \{(1,2)\ (2,1)\ (1,3)\ (3,1)\ (1,4)\ (4,1)\}$

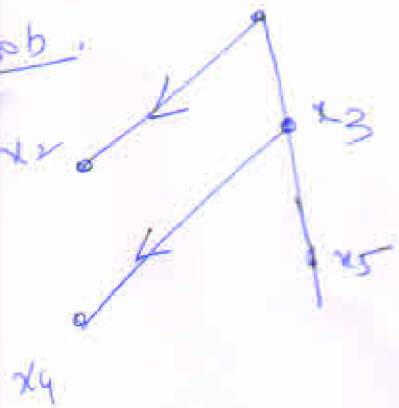
$$M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Relation is not reflexive. ∴ loops are not present in graph

It is symmetric be $M_R = (M_R)^T$

It is not transitive, not anti-symmetric

Prob.



$R = \{(x_1, x_2)\ (x_1, x_3)(x_3, x_4), (x_3, x_5)\}$

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Relation is not reflexive. loops are not present in graph

It is not symmetric, $M_R \neq (M_R)^T$

It is not transitive.

The relation is anti-symmetric. no direct relation path exit.

Partial ordering Relations

A relation R on a set P is called partial order relation or a partial ordering in P iff R is reflexive, antisymmetric & transitive. We denote a partial ordering by the symbol $\leq$.

A set P on which a partial ordering $\leq$ is defined is called a partially ordered set (or) a Poset. It is denoted by $(P, \leq)$ or $[P, \leq]$. The characteristic properties of a partial order can be described as follows.

1) $a \in A$, $a \leq a$ (reflexive)
2) $\forall a, b \in A$, if $a \leq b$ & $b \leq a$ then $a = b$ (Antisymmetry)
3) $\forall a, b, c \in A$ if $a \leq b$, $b \leq c$ then $a \leq c$ (transitive)

Example:- $[Z, <]$ is not a Poset because $<$ is not reflexive. $[Z, \leq]$ is a Poset.

Example:- Let A be any set & P(A) be the collection of all subsets of A. Then $[P(A), \subseteq]$ is a Poset.

$A = [a, b]$ $\quad P(A] = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

Let $[P, \leq]$ be a Poset. Elements a, b in P are said to be Comparable if either $a \leq b$ or $b \leq a$. otherwise they are incomparable.

~~Let~~ P

Totally ordered set :- Let $(P, \leq)$ be a Poset. If every Pair of elements of A are comparable, then $(P, \leq)$ is called a totally ordered set (or) chain (or) Simply ordered set.

Here the relation $\leq$ is called a total order (or) linear order (or) a simple Order on P.

HASSE DIAGRAM (OR) POSET DIAGRAM.

A Partial ordering $\leq$ on a set P Can be represented by means of a diagram known as HASSE diagram (or) Poset diagram of $(P, \leq)$

The procedure for drawing Hasse diagram for a poset P as follows.

1) Each Element is represented by a small circle or a dot.

(2) The circle for $a \in P$ is drawn below the circle for $b \in P$ if $a < b$.

(3) A line is drawn between a & b if b covers a i.e $a < b$.

if b does not cover a, then a & b are not Connected directly by a single line.

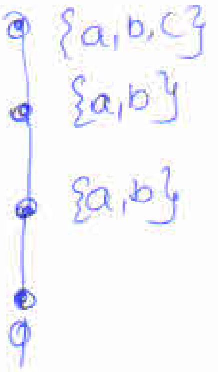
For a totally ordered set $(P, \leq)$ the Hasse diagram Consists of circles one below the other. Thus Poset is Called a chain.

Example :- Let $P_1 = [\phi, \{a\}, \{b\}, \{a, b, c\}]$

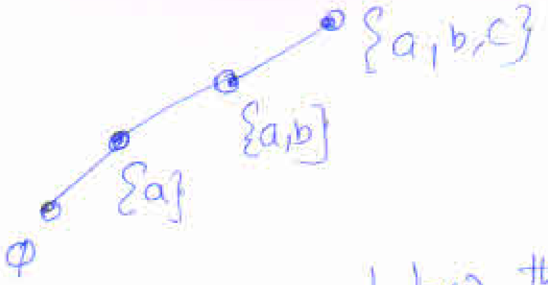
$P_2 = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}]$

P_1 is not a totally ordered set but P_2 is a Poset.

$\therefore \phi \subseteq \{a\} \subseteq \{a, b\} \subseteq \{a, b, c\}$.



(or)



Note The circle for $x \in P$ is draw below the circle for $y \in P$ if $x < y$ & a line is drawn between x & y if y cover x.

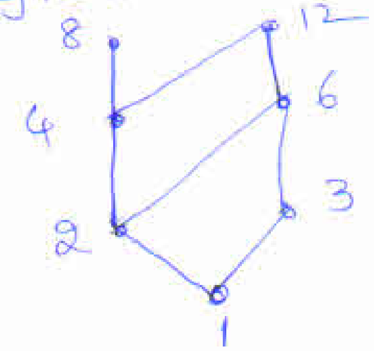
If $x < y$ but y does not cover x, then x & y are not connected directly by a single line.

1) **Problem** :- Draw the Hasse diagram representing the Partial Ordering $\{(a,b) / a \text{ divides } b\}$ on $\{1,2,3,4,6,8,12\}$

Let $P = \{1,2,3,4,6,8,12\}$

$$(P, \leq) = \{(1,2)(1,3)(1,4)(1,6)(1,8)(1,12)(2,4)(2,6)(2,8)(2,12)(3,6)(3,12)(4,8)(4,12)(6,12)\}$$

Hasse diagram of $\{(1,2,3,4,6,8,12), /\}$ is

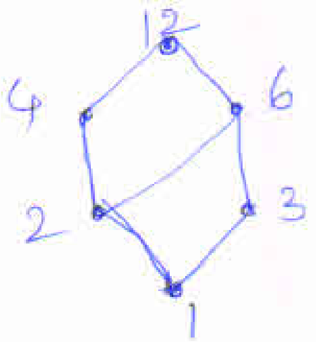


③

Draw Hasse diagram of Poset $(D_{12}, /)$

Sol:- $D_{12} = \{1, 2, 3, 4, 6, 12\}$

$= \{(1,2)(1,3)(1,4)(1,6)(1,12)(2,4)(2,6)(2,12)$
$(3,6)(3,12)(4,12)(6,12)\}$



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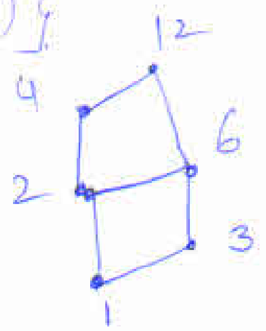
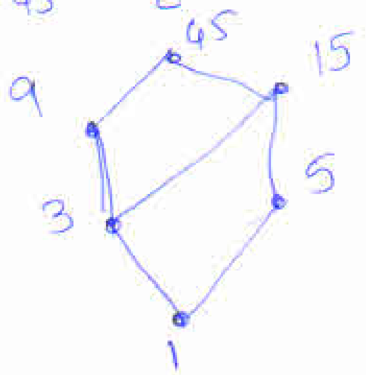
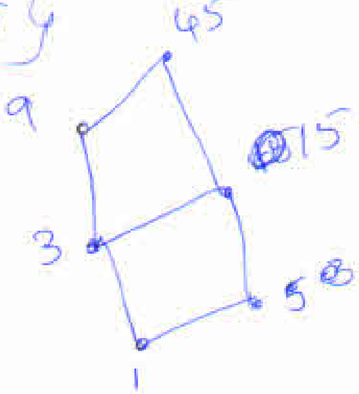


Problem. Draw the Hasse diagram of Poset $(D_{45}, /)$

Sol: $D_{45} = \{1, 3, 5, 9, 15, 45\}$

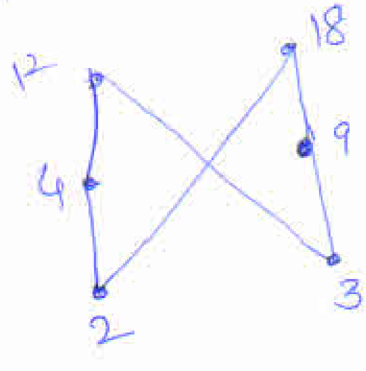


(or)



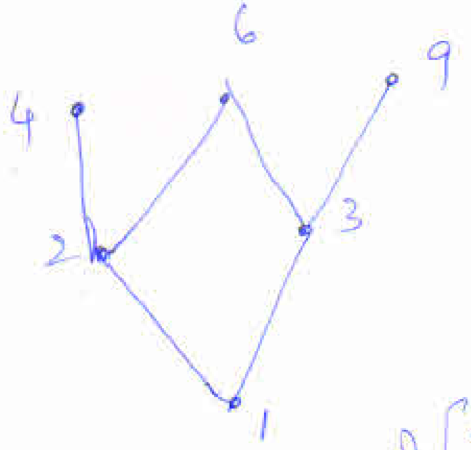
Problem. Draw the Hasse diagram of $\{\{2, 3, 4, 9, 12, 18\}, /\}$.

Sol: $(2,4)(2,12)(2,18)(3,9)(3,12)(3,18)(4,12)(9,18)$



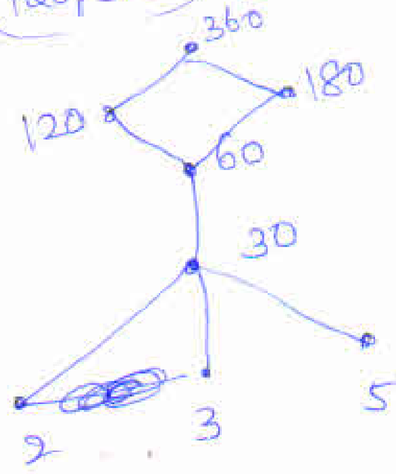
Problem. Draw Hasse diagram of [{1, 2, 3, 4, 6, 9}; /]

Sol: (1, 2) (1, 3) (1, 4) (1, 6) (1, 9) (2, 4) (2, 6) (3, 6) (3, 9)



Problem. Draw Hasse diagram of [{2, 3, 5, 30, 60, 120, 180, 360}; /]

Sol: (2, 30) (2, 60) (2, 120) (2, 180) (2, 360) (3, 30) (3, 60) (3, 120) (3, 180)
(3, 360) (30, 60) (30, 120) (30, 180) (30, 360) (60, 120) (60, 180)
(60, 360) (120, 360) (180, 360)

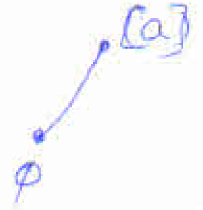


Problem. Let A be any finite set & $P(A)$ be the Power set of A, $\subseteq$ be the inclusion relation. On the elements of $P(A)$, Draw Hasse diagram of $(P(A), \subseteq)$ for

(i) $A = \{a\}$

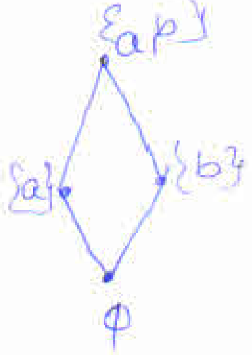
$P(A) = \{ \phi, \{a\} \}$.

$[P(A), \subseteq\}$ is a Poset



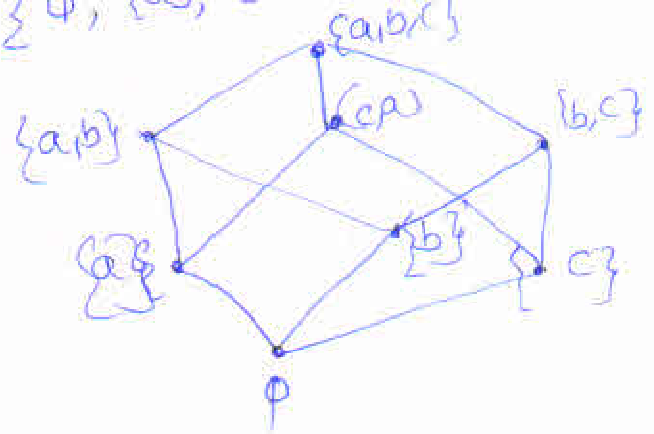
(ii) $A = \{a, b\}$

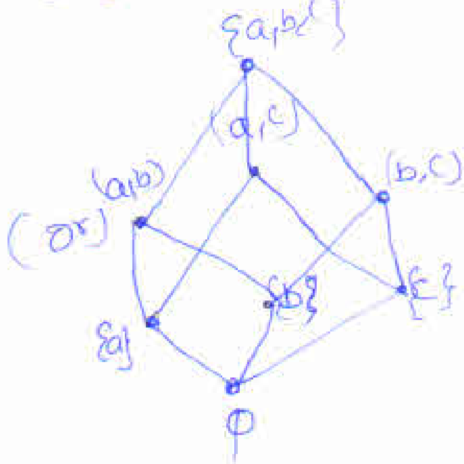
$P(A) = \{ \phi, \{a\}, \{b\}, \{a, b\} \}$



(iii) $A = \{a, b, c\}$

$P(A) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\} \}$



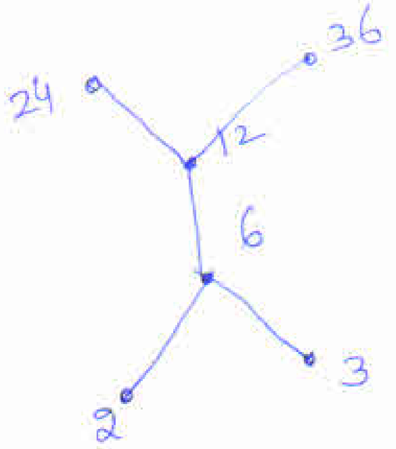


Problem. If $X = \{2, 3, 6, 12, 24, 36\}$ & the relation $\leq$ be such that $x \leq y$ if x divides y.

Draw the Hasse diagram of $\{X, \leq\}$

Sol: (2,6) (2,12) (2,24) (2,36) (3,6) (3,12) (3,24) (3,36)

(6,12) (6,24) (6,36) (12,24) (12,36)

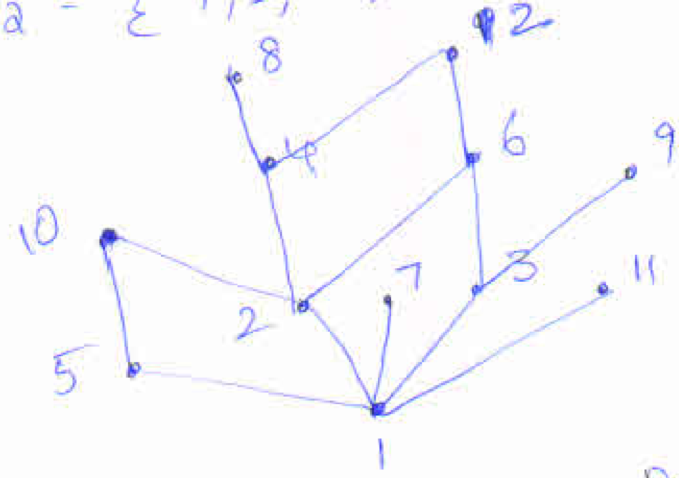


Note: on a Poset diagram there is a vertex for each element of A, all loops are omitted eliminating explicit representation of the reflexive property. An edge is not present in a Poset diagram if it is implied by the transitivity of the relation.

If we write $x < y$ to mean $x \leq y$ but $x \neq y$, then an edge connects a vertex x to a vertex y iff y covers x, i.e iff there is no other element $z \ni x < z$ & $z < y$.

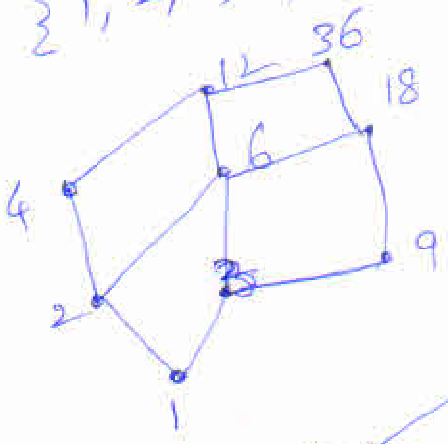
Problem Let I_{12} is the set of all positive integers which are less than or equal to 12. Draw the Hasse diagram of $(I_{12}, |)$

Sol. $I_{12} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

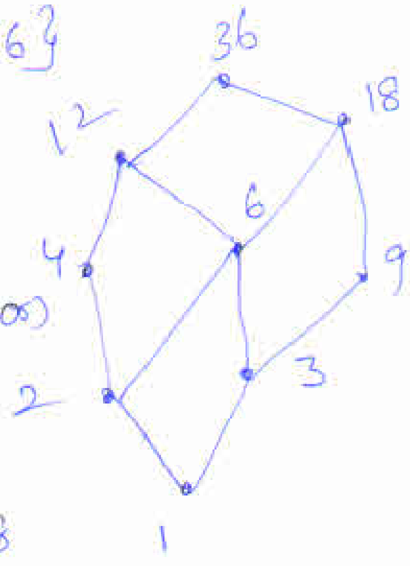


Problem:- Draw the Hasse diagram representing the positive divisors of 36.

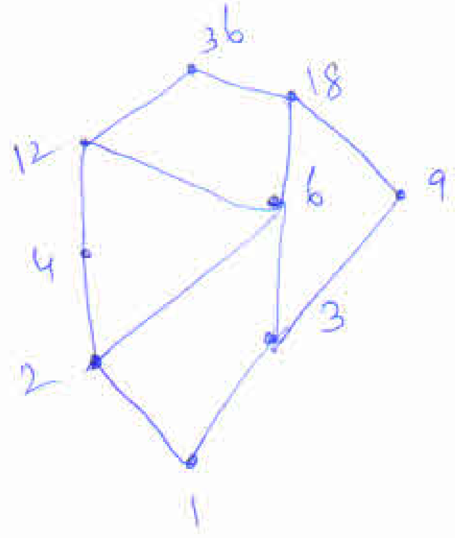
$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



(or)



(or)



Problem. Let $X = \{2, 3, 6, 12, 24, 36\}$ then prove $(X, /)$ is a Poset draw its Hasse diagram.

Let $a \in X$, $a/a \Rightarrow /$ is reflexive on X.

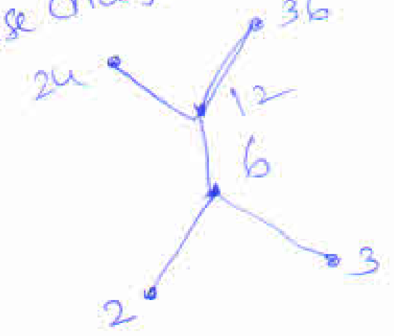
Let $a, b \in X$, $a|b$, $b|a \Rightarrow a = b$.

$\Rightarrow /$ is anti symmetric on X

Let $a, b, c \in X$, $a|b$, $b|c \Rightarrow a|c$.

$\Rightarrow /$ is transitive on X.

Its Hasse diagram is



Let $(P, \leq)$ be a Poset, & $A \leq P$, then $a \in A$ is Called a lower bound of A if $a \leq x \ \forall x \in A$ & if there are no lower bounds of A which are greater than "a", then "a" is called greatest lower bound (g.l.b) of A (or) infimum of A.

Let $(P, \leq)$ be a Poset, $A \leq P$, then $a \in A$ is called an upper bound of A if $x \leq a \ \forall x \in A$ and if there are no upper bound of A which are less than 'a' then a is Called least upper bound (l.u.b) of A (or) Supremum (sup) of A.

An element $a \in A$ is Called an upper bound of a subset B of A if $xRa \ \forall x \in B$.

An element $a \in A$ is Called a lower bound of a Subset B of A if $aRx \ \forall x \in B$.

An element $a \in A$ is Called a least upper bound (l.u.b) of a subset B of A if the following 2 Conditions hold:

(i) a is an upper bound of B.

(ii) If a is an upper bound of B then aRa.

A least upper bound is also Called as Supremum written as "sup"

An element $a \in A$ is called a greatest lower bound (G.l.b) of a subset B of A if the following 2 conditions hold.

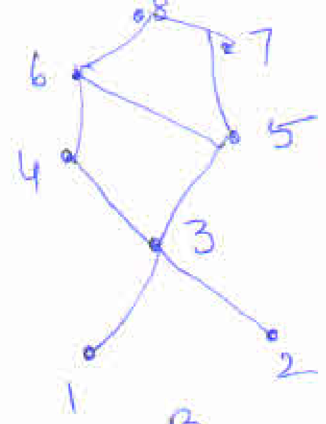
(i) a is a lower bound of B.

(ii) If a' is a lower bound of B then $a'Ra$.

A greatest lower bound is also called an Infimum written as "Inf"

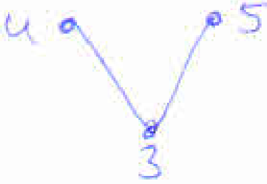
Note:- Every poset has atmost one greatest element & atmost one least element.

Example



B

B_1



$1R3$, $2R3$, ∴ 3 is an upper bound of B_1

Ily 4, 5, 6, 7, 8 are also upper bound of B_1

It has no greatest lower bound, it has l.u.b is 8.

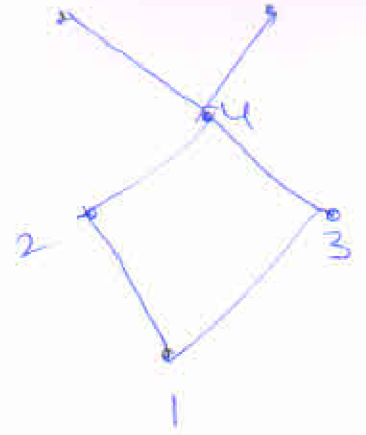
Special elements in Posets or External elements in posets.

(i) An element $a \in A$ is called a maximal element of A if whenever there is $x \in A \ni aRx$ then $x = a$.
This means that "a" is a maximal element of A iff in the Hasse diagram of R no edge starts at 'a'.

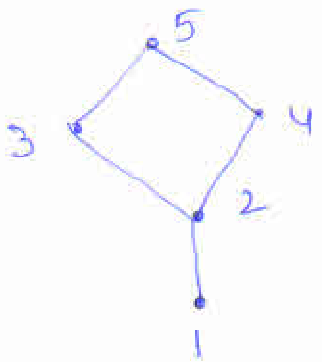
(ii) An element $a \in A$ is called a minimal element of A if whenever there is $x \in A \ni xRa$ then $x = a$.
This means that a is a minimal element of A iff in the Hasse diagram of R no edge terminates at 'a'.

(iii) An element $a \in A$ is called a greatest element of A if $xRa \ \forall x \in A$

(iv) An element $a \in A$ is called a least element of A if $aRx \ \forall x \in A$

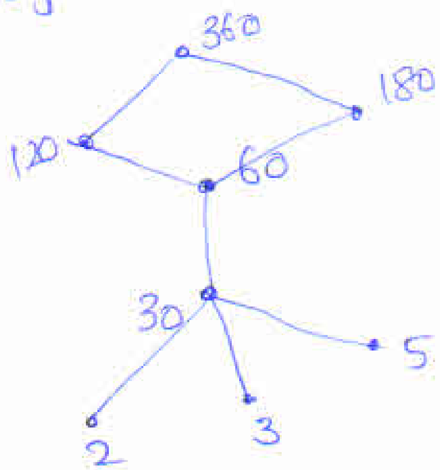


5 & 6 are Maximal elements
1 is a minimal element
1 is the least element.
There is no greatest element



5 is a Maximal as well as a greatest element
1 is a minimal as well as least element

<u>Problem</u> :- Consider the Poset { 2, 3, 5, 30, 60, 120, 180, 360 } having Hasse diagram shown below.



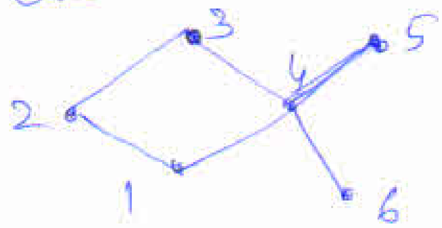
g.l.b of { 120, 180, 360 } is 60
The set { 2, 3, 5, 30 } has no lower bounds & no g.l.b
The set { 60, 120, 180, 360 } of the Poset has a minimal & least element is 60.

The set {120, 180, 360} has a minimal elements - 120 & 180 but no least element.

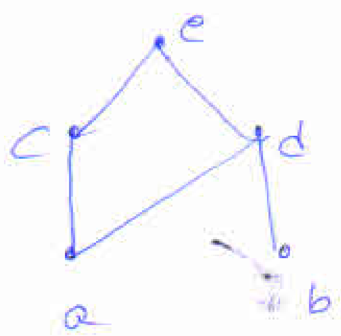
The set {120, 180, 360} has a g.l.b is 60 but the set {2, 3, 5, 30} has no lower bounds & hence no g.l.b

The set {60, 120, 180} has a l.u.b is 360

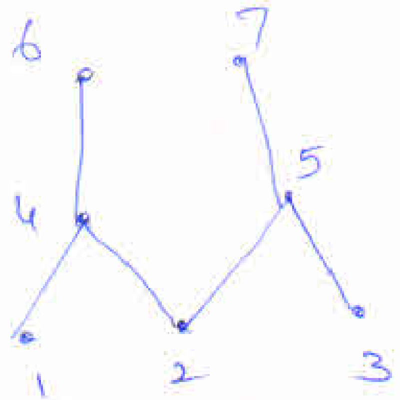
<u>Problem</u>. For the posets shown in the following Hasse diagrams, find all Maximal elements and all minimal elements:

1) 

Minimal elements - 1, 6
Maximal elements 3, 5

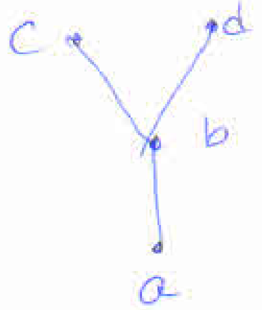
(2) 

Maximal element - e.
Minimal element - a, b

(3) 

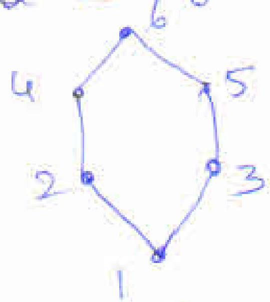
Maximal elements - 6, 7
Minimal element 1, 2, 3

4)

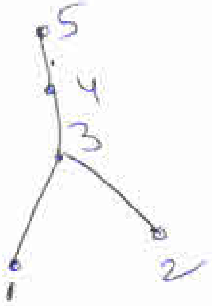


Maximal elements c, d
Minimal elements — a.

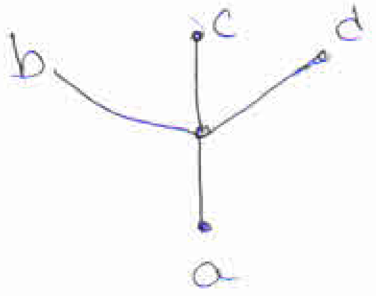
Problem. For the Posets Shown in the following Hasse diagram find the greatest & least elements.



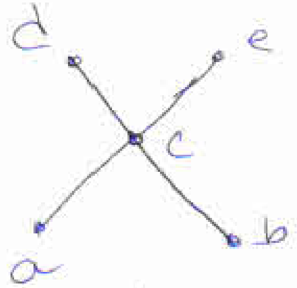
Greatest Element — 6
least element — 1



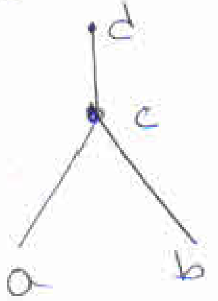
Greatest element - 5
least element — none



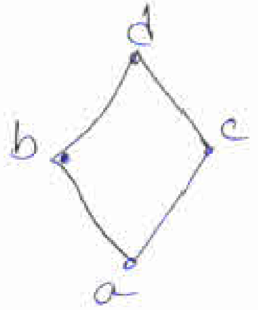
Greatest element - None
least element — a.



The poset has neither a least element nor a greatest element



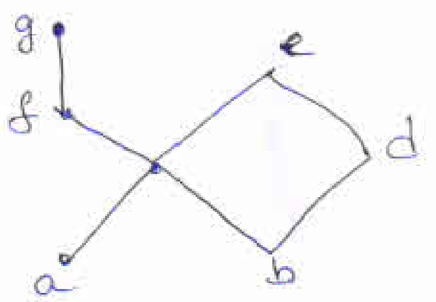
Greatest element - d
least element - None



Greatest element — d

Least element - a.

Problem Find the Maximal & minimal elements of the Poset A whose Hasse diagram is given below

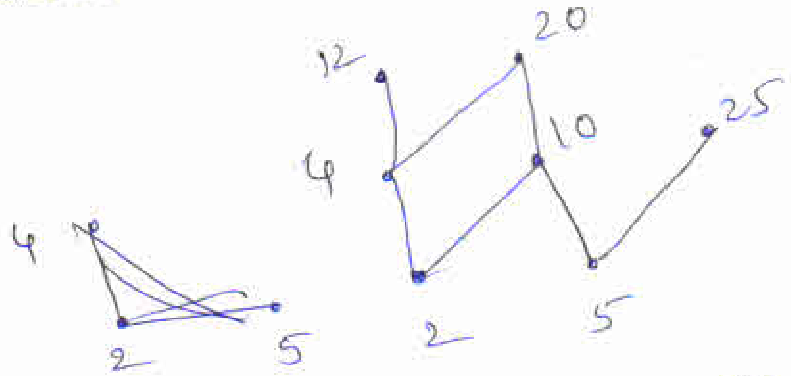


Maximal element g, e

Minimal element - a, b.

Problem which elements of the Posets {2, 4, 5, 10, 12, 20, 25}

are Maximal & Minimal.



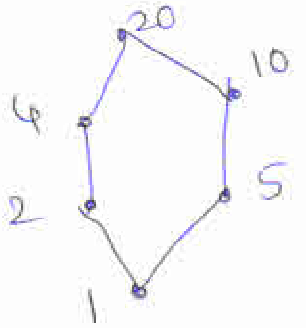
Maximal element - 12, 20, 25

Minimal element - 2, 5

Draw a poset diagram for each of the following posets & determine all maximal, minimal element. Greatest & least elements if they exist

Problem $(D_{20}, |)$

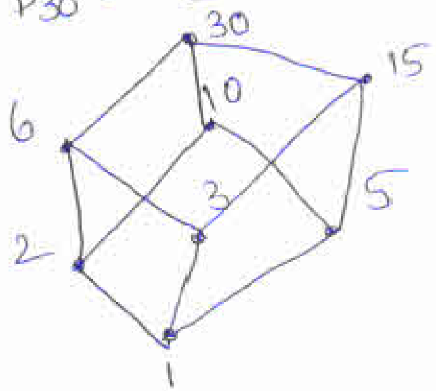
$D_{20} = \{1, 2, 4, 5, 10, 20\}$



Maximal element - 20
Minimal element - 1
Greatest element - 20
least element - 1

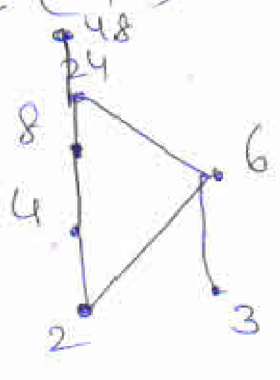
Problem $(D_{30}, |)$

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$



Maximal element - 30
Minimal element - 1
Greatest element - 30
least element - 1.

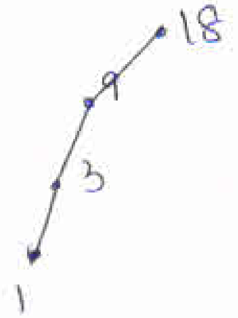
Problem $(A, |)$ where $A = \{2, 3, 4, 6, 8, 24, 48\}$



Maximal - 48
Minimal - 2, 3
Greatest - 48
least - does not Exist.

Prob: Draw the Hasse diagram of the following sets under the Partial ordering relation "divisibilites"

(i) $\{1, 3, 9, 18\}$

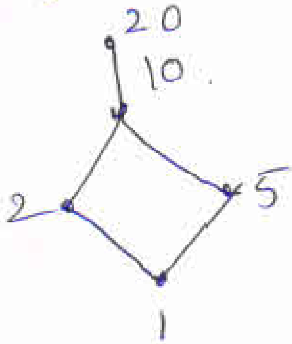


This is a totally ordered set in which the least Element is 1 & the Greatest element is 18.

(ii) $\{3, 5, 30\}$

There exists no least element in the Hasse diagram.

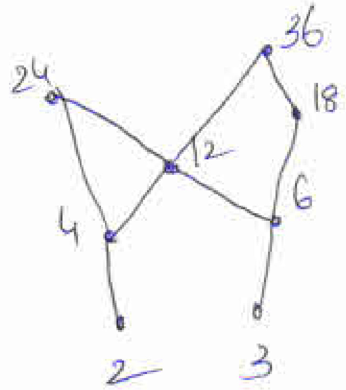
(iii) $\{1, 2, 5, 10, 20\}$



least element is 1

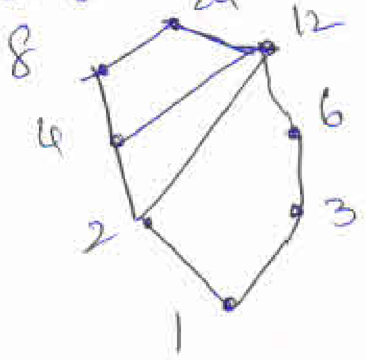
greatest element is 20

$(A, /)$ where $A = \{2, 3, 4, 6, 12, 18, 24, 36\}$.



Maximal 24, 36
Minimal – 2, 3
Greatest – does not Exist
least – does not Exist.

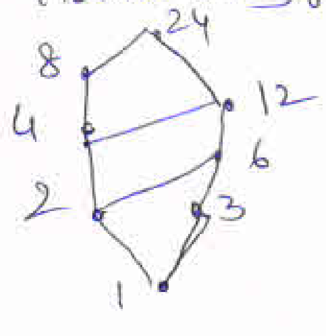
<u>Problem</u>: Let $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$ & relation divides "/" be a partial ordering on D_{24}. Then draw the Hasse diagram & $(D_{24}, /)$ & also find the following



1) all lower bounds of 8, 12 – 1, 2, 4
2) all upper bounds of 8, 12 – 24
3) g.l.b of 8, 12 – 1
4) l.u.b of 8, 12 – 24
5) greatest & least elements of this poset if exists
greatest – 24,
least – 1

<u>Prob</u>: Draw the Hasse diagram representing the positive integers of 24. Find Minimal, Maximal, Greatest and least Element

<u>Sol</u>: Hasse diagram $(D_{24}, /)$



24 is a Maximal as well as greatest Element

1 is a Minimal as well as least Element.

Lattice:- A Lattice is a Poset inwhich each pair of elements has a least upper bound (l.u.b) & greatest lower bound (g.l.b)

(or)

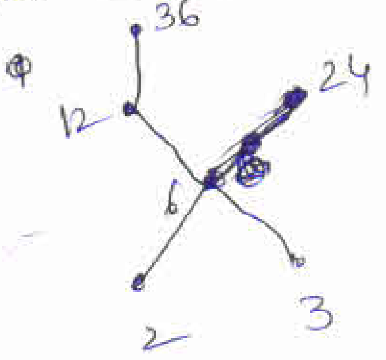
A Lattice is both a Join - semi lattice & a meet - semilattice.

we define a Join - semi lattice as a poset $[A, \subseteq]$ in which each pair of elements a & b of A have a l.u.b. we call this l.u.b the Join of a & b. It is denoted by $a \vee b$ (or) sum of a, b i.e $a \oplus b$

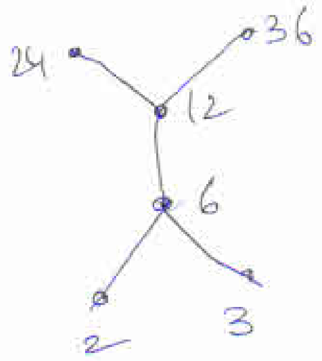
we define a meet - semilattice as a poset in which each pair of elements a & b have a g.l.b this g.l.b is called the meet of a & b, it is denoted by $a \wedge b$ (or) Product of a, b i.e $(a \times b)$

Note: Every lattice is a poset but all posets are not lattices.

Example :- $X = \{2, 3, 6, 12, 24, 36, 1\}$ is a Poset but not lattice. because g.l.b of $(2,3)$ is $1 \notin X$



(or)



Note:

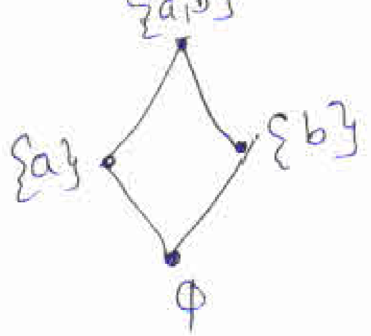
g.l.b $\{a,b\} = a \wedge b$ Called the meet of a & b

l.u.b $\{a,b\} = a \vee b$ Called the Join of a & b

Example :- Let $S = \{a, b\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

Then $[P(S), \subseteq]$ is a lattice because g.l.b & l.u.b of any Pair of elements of P(S) are belongs to P(S).



Operations on Relations

Let R & S be relation from a set A to set B. Then

(i) $R \cup S = \{(a,b) / (a,b) \in R \text{ or } (a,b) \in S\}$

(ii) $R \cap S = \{(a,b) / (a,b) \in R \;\&\; (a,b) \in S\}$

(iii) $R - S = \{(a,b) \in R - S, (a,b) \in R, (a,b) \notin S\}$

(iv) R' or $R^c = \{(a,b) \in R', (a,b) \notin R\}$.

Example Let $R = \{(1,3)(1,5)\}$

$S = \{(2,4)\ (1,5)\}$

$R \cup S = \{(1,3)(1,5)\ (2,4)\}$

$R \cap S = \{(1,5)\}$

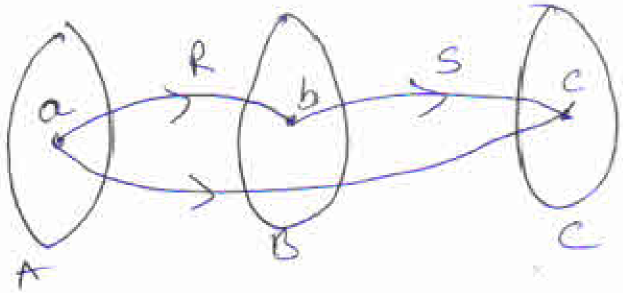
Inverse of R :- Let R be a relation from a set A to B. The inverse of R is relation from B to A & is given by

$R^{-1} = \{(y,x) / (x,y) \in R\}$

Let R be a relation from A to B & S be a relation from B to C. Then Composite relation of R & S is denoted by RoS (or) RS & it is defined as

$$RoS = \{(a,c) / a \in A, c \in C \ \exists\ b \in B \ni (a,b) \in R, (b,c) \in S\}$$

Composition of Relations.



Note
(i) $R \subseteq A \times B$ (ii) $S \subseteq B \times C$ (iii) $RoS \subseteq A \times C$.
(iv) $SoR \neq RoS$.

Prob Let R & S be two relations on $A = \{1,2\}$ & $R = \{(1,1)(2,2)\}$ $S = \{(1,2)(2,1)\}$.

Then $RoS = \{(1,2)(2,1)\}$

Problem

Let $A = \{a, b, c\}$ $B = \{1, 2, 3\}$ $C = \{x, y, z\}$

$R = \{(a,1)\ (a,3)\ (c,2)\ (a,2)\ (b,3)\}$

$S = \{(1,y)(1,z)(2,x)\ (2,z)\ (3,x)\ (3,y)\}$.

$R \circ S = \{(a,y)(a,x)\ (c,z)\ (a,z)(\ b,x)(b,y)\ (c,x)\}$.

$S \circ R = \{\ \}$ or ϕ

$R \circ S \neq S \circ R$.

Prob: Let R be two relations on $A = \{1, 2, 3\}$ &

$R = \{(1,1)\ (1,2)\ (2,3)\ (3,1)\ (3,3)\}$

$S = \{(1,2)\ (1,3)\ (2,1)\ (3,3)\}$ then find the following

$R \cup S = \{(1,2)\ (1,1)\ (2,3)\ (3,1)\ (3,3)\ (1,3)\ (2,1)\}$.

$R \cap S = \{(1,2)(3,3)\}$.

$R - S = \{(1,1)\ (2,3)\ (3,1)\}$

$R' = (A \times A) - R$.

$A \times A = \{1,2,3\} \times \{1,2,3\}$

$= \{(1,1)\ (1,2)\ (1,3)\ (2,1)\ (2,2)\ (2,3)\ (3,1)(3,2)(3,3)\}$

$R' = \{(1,3)(2,1)(3,2),(2,2)\}$.

$RoS = \{ (1,2) (1,1) (2,3) (3,2) \ (3,3) (1,3) \}$

$S^2 = SoS$.

$S = \{ (1,2) (1,3) (2,1) (3,3) \}$

$S = \{ (1,2) (1,3) (2,1) (3,3) \}$

$S^2 = \{ (1,1) (1,3) (2,2) (2,3) (3,3) \}$

$R^2 = RoR$.

$R = \{ (1,1) (1,2), (2,3), (3,1), (3,3) \}$

$R = \{ (1,1) (1,2), (2,3), (3,1), (3,3) \}$

$R = \{ (1,1), (1,2), (2,3), (3,1), (2,1), (1,2) (2,3) (3,2) \}$

$R^2 = \{ (1,1) (1,3), (3,3) (3,1), (2,1), (1,2) (2,3)(3,2) \}$

$\{ (1,1) (1,3), (2,1), (2,2), (3,1) (3,3) \}$

$SoR = \{ (1,3) (1,1), (1,3), (2,1), (2,2), (3,1) (3,3) \}$

$S - R = \{ (1,3) (2,1) \}$

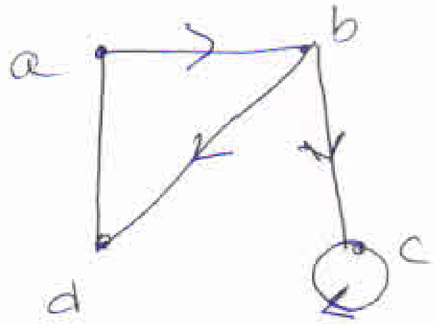
Problem: Consider the relation $R = \{ (a,b), (b,c), (b,d), (d,a) (c,c) \}$ on $A = \{a, b, c, d\}$.

(i) Draw a diagraph for the relation R.

(ii) Draw a diagraph for the Complement of R.

(iii) Draw a diagraph for the inverse of R.

(iv) Draw a diagraph for $R \cap R'$.

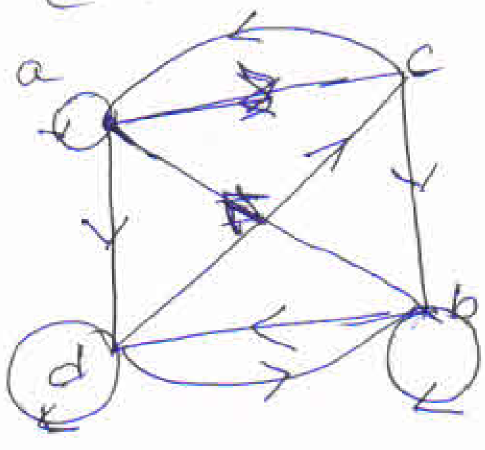


$R' = (A \times A) - R$

$= \{(a,b) \mid (a,b) \notin R\}$

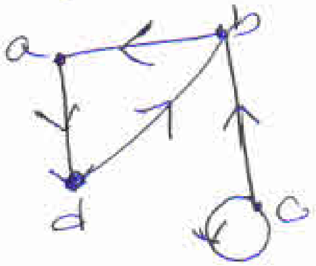
$A \times A = \{(a,a)\ (a,b),\ (a,c)\ (a,d)\ (b,a)\ (b,b)\ (b,c)\ (b,d)$
$(c,a)\ (c,b),\ (c,c)\ (c,d)\ (d,a)\ (d,b),\ (d,c),\ (d,d)\}$

$R' = \{(a,a)\ (a,c)\ (a,d),\ (b,a),\ (b,b)\ (c,a)\ (c,b)$
$(c,d)\ (d,b)\ (d,c)\ (d,d)\}$



(iii) $R = \{(a,b)\ (b,c)\ (b,d)\ (d,a)\ (c,c)\}$

$\bar{R}' = \{(b,a)\ (c,b),\ (d,b),\ (a,d),\ (c,c)\}$



(iv) $R \cap R^{-1} = \{(c,c)\}$

Suppose $R_1 \subseteq A \times B$, & $R_2 \subseteq B \times C$.
The composition of R_1 & R_2 is denoted as $R_1 R_2$ (or) $R_1 \circ R_2$
is defined as

$$R_1 R_2 = \{(x,z) \mid (x,y) \in R_1, (y,z) \in R_2\}.$$

<u>Example</u>: $R_1 \subseteq A \times A$ where $A = \{1,2,3\}$

Let $R_1 = \{(1,1)(1,3)(2,1)(2,3)(3,2)\}$

$R_1^2 = R_1 R_1$
$= \{(1,1)(1,3)(2,1)(2,3)(3,2)\}$
$\cdot \{(1,1)(1,3)(2,1)(2,3)(3,2)\}$
$= \{(1,1)(1,3)(1,2)(2,1)(2,3)(2,2)(3,1),(3,3)\}$

$R_1^3 = R_1^2 . R_1$
$= \{(1,2)(1,1)(1,3)(2,1)(2,3)(3,1)(3,3)(3,2)\}$

Problem :- If A is a set with m elements and B is a set ~~of~~ with n Elements. Find the number of relations from A to B.

Sol: Since a relation from A to B is precisely a subset of $A \times B$ the set of all relations from A to B is precisely the set of all subsets of $A \times B$. Therefore the number of relations from A to B is equal to the number of subsets of $A \times B$

Since $n(A) = m$ and $n(B) = n$ we have $n(A \times B) = mn$.

Therefore $A \times B$ has 2^{mn} number of subsets.

This implies that there are 2^{mn} relations from A to B

Problem: Let A, B are sets with $n(A) = 2$ and $n(B) = 3$ then the number of relations from A to B are $\underline{2^6 = 64}$.

Problem. Let $n(A) = m$, $n(B) = n$ there are 2^{mn} relations. from A to B we have

$$2^{3m} = 32768$$
$$= 8 \times 64 \times 64$$
$$= 2^{15}$$
$$3m = 15$$
$$\boxed{m = 5}$$

(or) $3^m \log_e 2 = \log(32768)$

$$m = \frac{\log_e 32768}{3 \log_e 2} = 5$$

Transitive closure

Let M_R be the relation matrix of a Relation on a set A of n elements then the transitive closure matrix $M_{R^+} = M_{R^1} \cup M_{R^2} \cup M_{R^3} \cup \cdots \cup M_{R^n}$.

If A is any finite set containing n elements and R is a relation on A then

$$R^+ = R_1 \cup R^2 \cup R^3 \cup \cdots \cup R^n.$$

Where R^+ is called transitive closure.

$$R^2 = R \cdot R$$
$$R^3 = R^2 \cdot R$$
$$\vdots$$
$$R^n = R^{n-1} R$$

Problem: Find the Boolean matrix representation of the transitive closure R^+ where $R = \{(a,b)(b,c)(d,c)(d,a)\}$

$$R^2 = R \cdot R = \{(a,b)(b,c)(d,c)(d,a)\} \cdot \{(a,b)(b,c)(d,c)(d,a)\}$$
$$= \{(a,c)(d,b)\}$$

$$R^3 = R^2 \cdot R = \{(a,c)(d,b)\} \cdot \{(a,b)(b,c)(d,c)(d,a)\}$$
$$= \{(d,c)\}$$

$$R^4 = R^3 \cdot R = \{(d,c)\} \cdot \{(a,b)(b,c)(d,c)(d,a)\}$$
$$= \{\phi\}$$

$R^+ = R \cup R^2 \cup R^3 \cup R^4$.

$= \{(a,b)(b,c)(d,c)(d,a)(d,b)(a,c)(d,c)\}$.

	a	b	c	d
a	0	1	1	0
b	0	0	1	0
c	0	0	0	0
d	1	1	1	0

Problem

Q. Let $A = \{a, b, c, d, e\}$
$R = \{(a,a)\ (a,b)\ (b,c)(c,d)\ (d,e)\ (c,e)\}$
find the transitive closure of R.

Sol: $R = \{(a,a)\ (a,b)(b,c)(c,d)(d,e)(c,e)\}$

$R^2 = R.R = \{(a,a)(a,b)(b,c)(c,d)\ (d,e)(c,e)\}.\{(a,a)(a,b)(b,c)(c,d)\ (d,e)(c,e)\}$

$= \{(a,b)\ (a,c)(b,d)\ (b,e)\ (c,e)\ (a,a)\}$

$R^3 = R^2 \cdot R = \{(a,b)(a,c)(b,d)\ (b,e)\ (c,e)\ (a,a)\}.\{(a,a)(a,b)(b,c)(c,d)(d,e)(c,e)\}$

$= \{(a,c)\ (a,d)(a,e)\ (b,e)\ (a,b)\ (a,a)\}$

$R^4 = R^3 . R = \{(a,d)\ (a,b)\ (a,c)\ (a,e)(a,a)\}$

$R^5 = R^4 . R = \{(a,a)\ (a,b)\ (a,c)\ (a,d)\ (a,e)\}$.

$R^+ = R^1 \cup R^2 \cup R^3 \cup R^4 \cup R^5$

$= \{(a,a)(a,b)(b,c)(c,d)(d,e)(c,e)(a,c)(b,d)(b,e)$
$(a,d)(a,e)\}$

	a	b	c	d	e
a	1	1	1	1	1
b	0	0	1	1	1
c	0	0	0	1	1
d	0	0	0	0	1
e	0	0	0	0	0

Prob :- Let $A = \{(1,2,3,4)\}$ and $R = \{(1,2)(2,3)(3,4)\}$ be a relation on A. find the transitive closure of R?

$R = \{(1,2)(2,3)(3,4)\}$

$R^2 = R.R = \{(1,2)(2,3)(3,4)\}.\{(1,2)(2,3)(3,4)\}$

$= \{(1,3)(2,4)\}$ ✓

$R^3 = R^2.R = \{(1,4)\}$

$R^4 = R^3.R = \{\}$ or ϕ.

$R^+ = R \cup R^2 \cup R^3 \cup R^4$

$= \{(1,2)(2,3)(3,4)(1,3)(2,4)(1,4)\}$

~~R×R~~ $R \times R$

$(1,2) \to (2,3)$

$(1,3)$

$(2,3) \to (3,4)$

$(2,4)$

$A \to B$ & $B \to C$

$A \to C$

	1	2	3	4
1	0	1	1	1
2	0	0	1	1
3	0	0	0	1
4	0	0	0	0

Note :- The transitive reflexive closure of R is denoted by R^* is defined as $R^* = R^+ \cup \{(a,a) \mid a \in A\}$

Problem :- (i) Given the adjacency matrix of the diagraph $G = \{(a, b, c, d), R\}$ where $R = \{(a,b)(b,c)(d,c)(d,a)\}$

(ii) Given the Boolean matrix representation of the transitive closure.

(iii) Given the boolean matrix representation of the transitive closure.

(i)

	a	b	c	d
a	0	1	0	0
b	0	0	1	0
c	0	0	0	0
d	1	0	1	0

(ii) $R^2 = \{(a,b)(b,c)(d,c)(d,a)\} \cdot \{(a,b)(b,c)(d,c)(d,a)\}$

$= \{(a,c)(d,b)\}$

$R^3 = R^2 \cdot R = \{d, c\}$

$R^4 = R^3 \cdot R = \{\ \}$

$R^+ = R \cup R^2 \cup R^3 \cup R^4$

$= \{(a,b)(b,c)(d,c)(d,a)(a,c)(d,b)$

	a	b	c	d
a	0	1	1	0
b	0	0	1	0
c	0	0	0	0
d	1	1	1	0

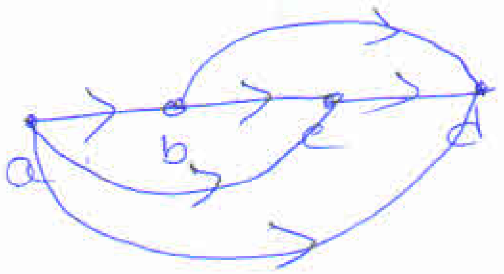
(iii) $R^* = R^+ \cup \{(a,a) \mid a \in A\}$

$= \{(a,b)(b,c)(d,c)(a,c)(d,a)(d,b)\} \cup \{(a,a)(b,b)(c,c)(d,d)\}$

$= \{(a,b)(b,c)(d,c)(a,c)(d,a)(d,b)(a,a)(b,b)(c,c)(d,d)\}$

	a	b	c	d
a	1	1	1	0
b	0	1	1	0
c	0	0	1	0
d	1	1	1	1

Problem: (i) Give the adjacency matrix of the diagraph. Show in the figure



(ii) Given the boolean matrix representation of the transitive closure of the relation represented by this diagraph and also find R^*.

Sol (i) $R = \{(a,b)\ (b,c)\ (c,d)\ (a,c)\ (a,d)\ (b,d)\}$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 1 \\ b & 0 & 0 & 1 & 1 \\ c & 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & 0 \end{array}$$

(ii) $R^2 = R \times R = \{(a,b)(b,c)(c,d)(a,c)(a,d)(b,d)\} \cdot \{(a,b)\ (b,c)\ (c,d)\ (a,c)\ (a,d)\ (b,d)\}$

$= \{(a,c)\ (b,d)\ (a,d)\}$

$R^3 = R^2 \cdot R = \{(a,c)\ (b,d)(a,d)\} \cdot \{(a,b)(b,c)\ (c,d)(a,c)\ (a,d)(b,d)\}$

$= \{a,d\}$

$R^4 = R^3 \cdot R = \{\ \}$

$R^{*} = R \cup R^2 \cup R^3 \cup R^4$

$= \{(a,b)\ (b,c)\ (c,d)\ (a,c)\ (a,d)\ (b,d)\}$

	a	b	c	d
a	0	1	1	1
b	0	0	1	1
c	0	0	0	1
d	0	0	0	0

$R = R^{*} = R^{+} \cup \{(a,a) \mid a \in A\}$

$= \{(a,b)\ (b,c)\ (c,d)\ (a,c)\ (a,d)\ (b,d)\} \cup \{(a,a)\ (b,b)\ (c,c)\ (d,d)\}$

=

	a	b	c	d
a	1	1	1	1
b	0	1	1	1
c	0	0	1	1
d	0	0	0	1

Warshall's Algorithm :-

Warshall's Algorithm is to find the transitive closure of a relation.

1) First transform to M_K all 1's in M_{K-1}

2) Record all positions $P_1 P_2 \dots P_n$ in column k of M_{K-1} where the entry is '1' and the positions $q_1, q_2, \dots q_n$ in a row k of M_{K-1} where the entry is 1.

(3) Put a '1' in each position (P_i, q_j) of M_K (Provided 1 is not already there from a previous step)

Problem: Find the transitive closure given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Sol.

M_0

	a	b	c	d
a	1	0	0	1
b	1	1	0	0
c	0	0	0	0
d	0	0	1	0.

I	Column C	Row R.
	(a, b)	(a, d)

Pairs (a, a) (a, d) (b, a) (b, d)

$$M_1 \quad \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 0 & 1 \\ b & 1 & 1 & 0 & 1 \\ c & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & 0 \end{array}$$

II C R.

b (a, b, d)

Pairs (b, a) (b, b) (b, d)

$$M_2 \quad \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 0 & 1 \\ b & 1 & 1 & 0 & 1 \\ c & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & 0 \end{array}$$

III C R.

d 0.

∴ No Pair

$$M_3 \quad \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 1 & 1 \\ b & 1 & 1 & 1 & 1 \\ c & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & 0 \end{array}$$

IV C R.

(a, b) c.

(a, c) (b, c)

$$M_4 \quad \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 0 & 1 \\ b & 1 & 1 & 0 & 1 \\ c & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & 0 \end{array}$$

$R^+ = \{(a,a)\ (a,c)\ (a,d)\ (b,a)\ (b,b)\ (b,c)\ (b,d)\ (d,c)\}$

(or)

$R = \{(a,a)(a,d)(b,a)(b,b)(d,c)\}$

$R^2 = \{(a,a)(a,d)(a,c)(b,a)(b,b)(b,c)\}$

$R^3 = R^2 \cdot R = \{(a,a)(a,d)(b,a)(a,c)(b,b)(b,d)\}$

$R^+ = R \cup R^2 \cup R^3$

$= \{(a,a)(a,c)(a,d)(b,a)(b,b)(b,c)(b,d)(d,c)\}$

Problem: Find the adJacency Matrix of transitive closure by warshall's algorithm by $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Sol

M_0

	a	b	c	d
a	0	1	0	0
b	1	0	1	1
c	0	0	0	1
d	0	0	0	0

I Column Row

b b.

Pair (b,b)

M_1

	a	b	c	d
a	0	1	0	0
b	1	1	1	0
c	0	0	0	1
d	0	0	0	0

II Column row

(a, b) (a, b, c)

(a,a) (a,b) (a,c)

(b,a) (b,b) (b,c)

M_2	a	b	c	d
a	1	1	1	0
b	1	1	1	0
c	0	0	0	1
d	0	0	0	0

III Column Row

(a, b) d.

Pairs ~~(c, d)~~

(a, d) (b, d)

M_3	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	0	0	0	1
d	0	0	0	0

IV Column R.

(a, b, c) -

No Pair.

M_4	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	0	0	0	1
d	0	0	0	0

$R^+ = \{(a,a)(a,b)(a,c)(a,d)(b,a)(b,b)(b,c)(b,d)(c,d)\}$

Problem: By using the warshall's algorithm Computer the adjacency matrix of the transitive closure of the digraph $a = \{(a,b,c,d,e), R\}$

$R = \{(a,b)\ (b,c)\ (c,d)\ (d,e)\ (e,d)\}$

M_0 :

	a	b	c	d	e
a	0	1	0	0	0
b	0	0	1	0	0
c	0	0	0	1	0
d	0	0	0	0	1
e	0	0	0	1	0

I

C	R
–	b

No Pair

M_1 =

	a	b	c	d	e
a	0	1	0	0	0
b	0	0	1	0	0
c	0	0	0	1	0
d	0	0	0	0	1
e	0	0	0	1	0

II

C	R
a	c

(a,c)

M_2 =

	a	b	c	d	e
a	0	1	1	0	0
b	0	0	1	0	0
c	0	0	0	1	0
d	0	0	0	0	1
e	0	0	0	1	0

III

C	R
(a,b)	d

(b,d) (a,d)

M_3 =

	a	b	c	d	e
a	0	1	1	1	0
b	0	0	1	1	0
c	0	0	0	1	0
d	0	0	0	0	1
e	0	0	0	1	0

IV

C	R
(b,d)	e
(c,a)	

(b,e) (d,e)

(c,e) (a,e)

$$M_4 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

V e R

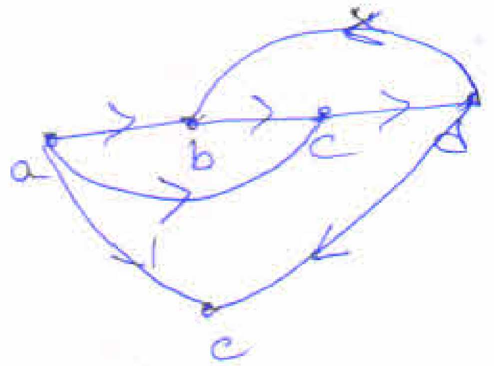
(b, c, d) a d

(b, d) (e, a) (d, d) (a, d)

$$M_5 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$R^+ = \{(a,b)\,(b,c)\,(b,d)\,(b,e)\,(c,d)\,(c,e)\,(d,d)\,(d,e)\,(e,d)\}$

Problem: By using the warshall's algorithm compute the transitive closure of the adjacency matrix of the diagraph.



Sol

$M_0 =$

	a	b	c	d	e
a	0	1	1	0	0
b	0	0	1	0	0
c	0	0	0	1	0
d	0	1	0	0	1
e	1	0	0	0	0

Ist column row

e (b, c)

Pair (e, b) (e, c)

$$M_1 = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 1 & 0 & 0 \\ b & 0 & 0 & 1 & 0 & 0 \\ c & 0 & 0 & 0 & 1 & 0 \\ d & 0 & 1 & 0 & 0 & 1 \\ e & 1 & 1 & 1 & 0 & 0 \end{array}$$

II Column Row

(a, d, e) c.

(a, c) (d, e) (e, c)

$$M_2 = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 1 & 0 & 0 \\ b & 0 & 0 & 1 & 0 & 0 \\ c & 0 & 0 & 0 & 1 & 0 \\ d & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 1 & 1 & 0 & 0 \end{array}$$

III Column Row

(a, b, d, e) d

(a, d) (b, d) (d, d) (e, d)

$$M_3 = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 1 & 1 & 0 \\ b & 0 & 0 & 1 & 1 & 0 \\ c & 0 & 0 & 0 & 1 & 0 \\ d & 0 & 1 & 1 & 1 & 1 \\ e & 1 & 1 & 1 & 1 & 0 \end{array}$$

IV Column Row

(a, b, c, d, e) (b, c, d, e)

(a, b) (a, c) (a, d) (a, e)
(b, b) (b, c) (b, d) (b, e)
(c, b) (c, c) (c, d) (c, e)
(d, b) (d, c) (d, d) (d, e)
(e, b) (e, c) (e, d) (e, e)

$$M_4 = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 1 & 1 & 1 \\ b & 0 & 1 & 1 & 1 & 1 \\ c & 0 & 1 & 1 & 1 & 1 \\ d & 0 & 1 & 1 & 1 & 1 \\ e & 0 & 1 & 1 & 1 & 1 \end{array}$$

V Column Row

(a, b, c, d, e) (b, c, d, e)

Problems. using warshall's algorithm Compute the adjacency matrix of the transitive closure of the diagraph $G = \{a, b, c, d, e\}$

$R = \{(a,b)\ (b,c)\ (c,d)\ (d,e)\ (e,d)\}$

(i) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.