RECURRENCE RELATIONS

Definition:-

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, a_2, \ldots, a_{n-1}$, for all integers $n \geq 1$,

Note:- Recurrence relation is also called difference equation.

Example:-

1) If a_n denotes the sum of the first 'n' positive integers, then $S_n = n + S_{n-1}$ which is a recurrence relation.

2) If S_n denotes the n^{th} term of a geometric progression with common ratio r, then $S_n = r \cdot S_{n-1}$ which is a recurrence relation.

3) $a_n - 3a_{n-1} + 2a_{n-2} = 0$

4) $a_n - 5a_{n-1} + 6a_{n-2} = n^2 + 1$

5) $a_n^2 + a_{n-1}^2 = -1$

6) $a_n - (n-1)(a_{n-1}) + (n-1)a_{n-2} = 0$.

7) $a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 5n$.

8) $a_n = d_0 a_{n-1} + d_1 a_{n-2} + \cdots + d_{n-1} a_0$

Suppose n & k are non-negative integers. A recurrence relation of the form $c_0(n)a_n + c_1(n)a_{n-1} + \cdots c_k(n)a_{n-k} = f(n)$ → (1)

for $n \geq k$, where $c_0(n), c_1(n) \ldots, c_k(n)$ & $f(n)$ are functions of n is said to be a linear recurrence relation.

If $c_0(n)$ & $c_k(n)$ are not identically zero, then it is said to be a linear recurrence relation of degree k.

If $c_0(n), c_1(n), \ldots c_k(n)$ are constants, then the recurrence relation is known as a linear recurrence relation with constant coefficients.

If $f(n) = 0$, then (1) is said to be homogeneous recurrence relation.

If $f(n) \neq 0$, then (1) is said to be inhomogeneous recurrence relation.

Note:-

All the examples above are linear recurrence relations except (5) & (8) because relation (5) contains square term., (8) is not in recurrence relation.

The relations in ①, ②, ③, ④ ; ⑦ are linear with constant coefficients.

Relations ①, ② have degree 1, ③ & ④ have degree 2, ⑦ have degree 3.

Relations 2, 3, 6 are homogeneous

Relations 1, 4, 7 are inhomogeneous.

Solutions of Recurrence Relations.

Definition:- A sequence $\{a_n\}_{n=0}^{\infty}$ is said to be a solution of a recurrence relation if each value a_n i.e $a_0, a_1, a_2, \ldots a_n \ldots$ satisfies the recurrence relation.

Example:-
$\{a_n\}_{n=0}^{\infty}$ where $a_n = 2^n$ is the solution of the recurrence relation $a_n = 2a_{n-1}$, $n \geq 1$. & the sequence $\{c\,2^n\}_{n=0}^{\infty}$ where c is constant is also solution of recurrence relation $a_n = 2a_{n-1}$.

Three methods of solving Recurrence relations:

1) Substitution (Iteration)
2) Generating functions
3) characteristic roots.

Method of characteristic Roots :

In this, we see the method of solving homogeneous linear recurrence relation of degree 'k' by the method of characteristic roots.

Definition :

Let $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0, \; n \geq k, \; c_k \neq 0$ $\rightarrow (1)$.

be a linear homogeneous recurrence relation of degree k. Then the equation,

$$t^k + c_1 t^{k-1} + c_2 t^{k-2} + \cdots + c_k = 0 \rightarrow (2)$$

is called the characteristic equation of the given recurrence relation (1).

Degree of equation (2) is k, therefore it has 'k' roots.

Let $\alpha_1, \alpha_2, \ldots \alpha_k$ be the roots of the equation (2) & the roots $\alpha_1, \alpha_2, \ldots \alpha_k$ are called characteristic roots.

Case I :- If $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_k$ are roots of the characteristic equation such that $\alpha_1 \neq \alpha_2 \neq \cdots \neq \alpha_k$.

then solution is $\boxed{a_n = c_1 \alpha_1^n + c_2 \alpha_2^n + \cdots c_k \alpha_k^n}$

Case 2 :- If two roots are repeated i.e $\alpha_1 = \alpha_2$ then solution is $\boxed{a_n = (c_1 + c_2 n)\alpha_1^n}$.

Case 3 :- If three roots are repeated i.e $\alpha_1 = \alpha_2 = \alpha_3$ then solution is $\boxed{a_n = (c_1 + c_2 n + c_3 n^2)\alpha_1^n}$.

Case 4 :- If two roots are imaginary i.e $\alpha_1 = \alpha + i\beta$, $\alpha_2 = \alpha - i\beta$ then $\boxed{a_n = (c_1 \cos n\beta + c_2 \sin n\beta)\alpha^n}$

Case 5 :- If two complex roots are repeated i.e

$\alpha_1 = \alpha_2 = \alpha + i\beta$, $\alpha_3 = \alpha_4 = \alpha - i\beta$.

then solution is $\boxed{a_n = [(c_1 + c_2 n)\cos n\beta + (c_3 + c_4 n)\sin n\beta]\alpha^n}$.

Case 6 :- If the two roots are irrational

i.e $\alpha_1 = \alpha + \sqrt{\beta}$ $\alpha_2 = \alpha - \sqrt{\beta}$. then

$$\boxed{a_n = e^{\alpha n}(c_1 \cosh\sqrt{\beta}\, n + c_2 \sinh\sqrt{\beta}\, n)}.$$